Fixed Income and Credit Risk
Fall 2012
Solutions of the Midterm Exam exercises

November 05, 2012

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Exercise N° 01.

The data of the problem are : $A_t = 120000$, $R = 5\%$, $n = 8$.
i) If the annual rate R is compounded semi-annually ($m = 2$), the value of the principal after 8 years is:

$$V_F(2,8) = 120000 \times (1 + 0.05/2)^{2\times 8} = 178140.67\,,$$

and the effective annual rate is $R^{eff} = (1 + 0.05/2)^2 - 1 = 0.050625$.

ii) If the annual rate is compounded quarterly ($m = 4$), then:

$$V_F(4,8) = 120000 \times (1 + 0.05/4)^{4\times 8} = 178575.66\,,$$

and the effective annual rate is equal to $R^{eff} = (1 + 0.05/4)^4 - 1 = 0.050945$.

iii) $V_F(4,8) > V_F(2,8)$ because of the larger compounding frequency of the annual nominal rate in case ii) with respect to case i).

iv) The equivalent annual continuously compounded rate, associated to the annual rate compounded semi-annually, is $R_\infty = 2\ln(1 + 0.05/2) = 0.049385$. The equivalent annual continuously compounded rate, associated to the annual rate compounded quarterly, is $R_\infty = 4\ln(1 + 0.05/4) = 0.049690$.

Exercise N° 02.

The coupon bond price at date $t = 0$ is given by:

$$CB(0,T) = \sum_{i=1}^{T} (C_i) \times (1 + Y)^{-i}\,,$$

where C_i is the i^{th} cash flow (coupon and principal payments) and where Y denotes the yield-to-maturity (level of the flat term structure). The associated duration D is given by:

$$D = \frac{\displaystyle\sum_{i=1}^{T} (i \times C_i) \times (1 + Y)^{-i}}{\displaystyle\sum_{i=1}^{T} (C_i) \times (1 + Y)^{-i}}\,.$$

In our case, the numerator is given by $\displaystyle\sum_{i=1}^{3} (i \times 8) \times (1 + 0.1)^{-i} + (3 \times 100) \times (1 + 0.1)^{-3} = 263.92$.

The denominator, that is the bond price, is $\displaystyle\sum_{i=1}^{3} 8 \times (1 + 0.1)^{-i} + 100 \times (1 + 0.1)^{-3} = 95.03$. The duration of the bond is therefore $D = 263.92/95.03 = 2.78$ years.

The convexity κ associated to $CB(0,T)$ is:

$$\kappa = \frac{\sum_{i=1}^{T} \frac{(i \times (1+i) \times C_i)}{(1+Y)^{i+2}}}{\sum_{i=1}^{T} \frac{C_i}{(1+Y)^i}} = \sum_{i=1}^{T} \frac{(i \times (1+i) \times C_i)}{(1+Y)^{i+2}} / CB(0,T) \ .$$

In our case, the numerator is given by $\sum_{i=1}^{3} (i \times (1+i) \times 8) \times (1+0.1)^{-i-2} + (12 \times 100) \times (1+0.1)^{-5} =$ 849.52, while the denominator is the bond price $CB(0,T) = 95.03$.

This means that $\kappa = 849.52/95.03 = 8.94 \ (years)^2$.

Exercise N° 03.

i) We have seen during Lecture 1 that the bond price variation can be approximated in the following way:

$$\frac{\Delta CB(Y)}{CB(Y)} = -D_{mod} \Delta Y \ ,$$

$$\text{where } D_{mod} = \frac{D}{1+Y} \text{ is the modified duration} \, .$$

In our case we have:

$$\frac{\Delta CB(Y)}{CB(Y)} = -\frac{D}{1+Y} \Delta Y = -\frac{1.9429}{1.07} \times 0.01 = -1.816\% \, .$$

ii) We have also seen during Lecture 1 that the bond price variation can be approximated in the following way:

$$\frac{\Delta CB(Y)}{CB(Y)} = -D_{mod} \Delta Y + \frac{\kappa}{2} (\Delta Y)^2 \ ,$$

$$\text{where } D_{mod} = \frac{D}{1+Y} \ ,$$

$$\text{and } \kappa = \frac{1}{CB(Y)} \frac{d^2 CB(Y)}{dY^2} \text{ is the } \textit{convexity} \, .$$

In our case we have:

$$\frac{\Delta CB(Y)}{CB(Y)} = -\frac{D}{1+Y} \Delta Y + \frac{\kappa}{2} (\Delta Y)^2$$

$$= -\frac{1.9429}{1.07} \times 0.01 + \frac{5.0411}{2} \times (0.01)^2 = -1.816\% + 0.025\% = -1.791\% \, .$$

Given that the initial bond price is $CB(0,T) = 98.20$, the approximated new bond price, determined by the *"duration + convexity"* approximation is equal to 96.4412, the one determined by the *"duration"* approximation only is 96.4167 while, the exact new bond price is 96.4335. We thus observe that the second-order approximation provided by the duration and convexity terms is, by construction, more precise than the first-order approximation of the duration term only.

iii) If interest rates rise from 7% to 12%, we have that the first-order approximation of the bond price variation is

$$\frac{\Delta CB(Y)}{CB(Y)} = -\frac{1.9429}{1.07} \times 0.05 = -9.079\%,$$

while the second-order approximation is

$$\frac{\Delta CB(Y)}{CB(Y)} = -\frac{1.9429}{1.07} \times 0.05 + \frac{5.0411}{2} \times (0.05)^2 = -9.079\% + 0.63\% = -8.449\%.$$

This means that, the "duration"-based approximation provides as new price the value 89.2844, while the "duration+convexity"-based approximation fix as new bond price the value 89.9031. The exact new bond price is 89.8597.

The error generated by the duration-based approximation moves from 0.0174% (when $\Delta Y = 0.01$) to 0.6443% (when $\Delta Y = 0.05$) while, the one generated by the "duration+convexity"-based approximation moves from 0.0080% to 0.0483%. We observe that, given a large (non infinitesimal) yield variation, the approximation provided by the duration term only becomes quite poor (37 times larger) while the one provided by the duration and convexity terms remains "decent" even if the error increases (6 times larger). In other words, in presence of an infinitesimal yield variation, the first-order approximation (duration-based) is sufficiently precise, while in presence of a large yield variation the second-order approximation becomes necessary given the large errors produced by the former method.

Exercise N° 04.

i) LIBOR stands for London Interbank Offered Rate. It is the rate at which an individual Contributor Panel bank could borrow funds, were it to do so by asking for and then accepting inter-bank offers in reasonable market size, just prior to 11:00 London time.

EURIBOR stands for Euro Interbank Offered Rate. It is the rate at which Euro interbank term deposits within the Euro zone are offered by one Prime Bank to another Prime Bank. It is computed as an average of daily quotes provided for fifteen maturities by a panel of 43 of the most active Banks in the Euro zone. It is quoted on an act/360 day count convention, and is fixed at 11:00am [CET] displayed to three decimal places.

ii) The main differences between LIBOR and EURIBOR rates are the following:

- The LIBOR rates represents the bank's perception of its cost of funds in the inter-bank market, while banks in the EURIBOR panel are thus asked to submit rates that reflect the "best" lending rates in unsecured cash transactions that could take place in the euro area between the "best banks". EURIBOR rates are therefore, by definition, independent of the situation of the banks submitting those rates or of actual transactions in which they engage.

– The LIBOR rate is calculated for 10 different currencies across 15 maturities from overnight to 1 year, while the EURIBOR rate is calculated for Euro and US dollar only, for maturities from 1 week to 1 year.

– The LIBOR is calculated by the British Banker's Association (BBA) by eliminating the highest and the lowest 25% of the submitted rates and then averaging the remaining 50%. This average rate is the published (at around 11:45 London time) Libor rate. The EURIBOR is calculated by the European Banking Federation (EBF) as a trimmed average since the average of the contributions is calculated after eliminating the 15% highest and lowest contributions.

– The number of banks in the Contributor Panel of LIBOR is 18 maximum (for US Dollar), while the number of contributors at the EURIBOR panel (for Euro) is 43. Contributor banks for LIBOR rates are selected for currency panels in line with three principles: 1) scale of market activity, 2) credit rating and 3) perceived expertise in the currency concerned. Panel banks for EURIBOR rates are the banks with the highest volume of business in the euro area money markets and it is made up of: 1) banks from EU countries participating in the euro from the outset; 2) banks from EU countries not participating in the euro from the outset and; 3) large international banks from non-EU countries but with important euro zone operations.

Exercise N° 05.

The profit $\Pi(t,T)$ (say) of the trader is:

$$\begin{aligned}
\Pi(t,T) &= P_T - P_t - \left[P_t \times r \times \frac{n}{360}\right] \\
&= (CB_T^{clean} + AI_T) - (CB_t^{clean} + AI_t) \times \left[1 + r \times \frac{n}{360}\right] \\
&= CB_T - CB_t \times \left(1 + r \times \frac{n}{360}\right).
\end{aligned}$$

It is clear that $\Pi(t,T) = 0$ if and only if

$$\frac{CB_T - CB_t}{CB_t} = r \times \frac{n}{360}$$

where the RHS of the equation is the appropriate repo rate for the time interval (t,T). If

$$\frac{CB_T - CB_t}{CB_t} > r \times \frac{n}{360}$$

the trade provides a positive carry, while if we have

$$\frac{CB_T - CB_t}{CB_t} < r \times \frac{n}{360}$$

the trade provides a negative carry.

Exercise N° 06.

We have a financial market with, at date $t = 0$, the vector of asset prices $S(0) = [S_0(0), S_1(0)]' = [1/3, 1]'$, and at date $t = 1$ the payoff matrix:

$$\mathbf{S} = \begin{bmatrix} S_0(1,\omega_1) & S_0(1,\omega_2) & S_0(1,\omega_3) \\ S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 6 \end{bmatrix},$$

where $S_i(t,\omega_j)$ denotes the price at date t of the asset i under the j^{th} state of the world ($j \in \{1, 2, 3\}$).

i) The first fundamental theorem of asset pricing tell us that the financial market $\{S(0), \mathbf{S}\}$ is arbitrage-free if and only if there exists a vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ of state prices such that $S(0) = \mathbf{S}q^{ad}$.

In our case, we have to find a positive solution $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})'$ to the:

$$\begin{cases} \frac{1}{3} & = & q_1^{ad} + q_2^{ad} + q_3^{ad} \\ 1 & = & q_1^{ad} + 3q_2^{ad} + 6q_3^{ad}\,. \end{cases}$$

Now, taking q_3^{ad} as a parameter (we have two equations and three unknowns) the solution for q_1^{ad} and q_2^{ad} is:

$$\begin{cases} q_1^{ad} & = & \frac{3}{2}q_3^{ad} \\ q_2^{ad} & = & \frac{1}{3} - \frac{5}{2}\,q_3^{ad}\,. \end{cases}$$

and we have $q_1^{ad} > 0$ and $q_2^{ad} > 0$ if and only if $0 < q_3^{ad} < \frac{2}{15}$. This means that, for any positive value of $q_3^{ad} \in]\,0, \frac{2}{15}\,[$ we have a vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ solving the system $S(0) = \mathbf{S}q^{ad}$ and therefore the market is arbitrage-free.

ii) The second fundamental theorem of asset pricing tell us that a no-arbitrage market is complete if and only if there exists a unique vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ of state prices such that $S(0) = \mathbf{S}q^{ad}$. From question *i)* we have seen that we have an infinity of positive solutions q^{ad}, each one associated to a different value of $q_3^{ad} \in]\,0, \frac{2}{15}\,[$. This means that the market is incomplete.

iii) Given that the market is arbitrage-free and incomplete, we have an infinity of equivalent martingale measures. For any $q_3^{ad} \in]\,0, \frac{2}{15}\,[$, the probability measure $\mathbb{Q}$ given by (q_1, q_2, q_3), with $q_j = q_j^{ad}/\sum_{j=1}^{3} q_j^{ad}$ for all $j \in \{1, 2, 3\}$, is an equivalent martingale measure for the market. Indeed, from the risk-free asset we have that the (continuously compounded) short rate is $r = \ln(1/S_0(0))$ and therefore $e^r = 1/\sum_{j=1}^{3} q_j^{ad}$. Thus, we can represent the price of the risky asset as:

$$S_1(0) = \sum_{j=1}^{3} \frac{q_j^{(ad)}}{q_0^{(ad)}}\, q_0^{(ad)} S_1(1,\omega_j) = \sum_{j=1}^{3} e^{-r}\, q_j\, S_1(1,\omega_j) = E^{\mathbb{Q}}\left[e^{-r} S_1(1)\right],$$

proving the fact that $\mathbb{Q}$ is an equivalent martingale measure.