Fixed Income and Credit Risk: solutions to exercise sheet no. 10

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1. The key point here is that the default probability should be the same for both the USD and AUD issues, since the issuer of the debt is the same firm. We may also assume that the recovery in the two cases is the same. The math, then, comes down to calibrating a value λ for the USD issue in the Hull-White scheme, and then applying this value for pricing the AUD bond, along with the AUD values for the risk-free discount factors. We can then solve for the par coupon in the AUD case.

   The pricing is implemented in the solutions spreadsheet. The results for the three USD bond prices are as follows:

USD price	λ	USD YTM	AUD coupon
1.00	0.04415	4.00%	7.73%
0.95	0.06581	5.16%	8.84%
1.05	0.02457	2.91%	6.64%

   Note that since the AUD bond is always priced at par, the coupon is identical to the YTM. Observe that as the USD price changes from 100 to 95, the yield in USD changes by a greater amount (116bp) than the yield in AUD (111bp), even though we have applied the same change in default probabilities to both the USD and AUD pricing. This illustrates a weakness of using YTM-based spread measures to compare the valuation of risky credits across different currencies.

2. Let $s^\star$ denote the quoted fair spread, s denote the standard spread paid on the new contract and U denote the upfront payment required by the protection buyer in the new contract. Let $D(t$ denote the risk-free discount factor, that is, the present value of one risk-free currency unit to be paid at future time t. The fair spread is defined to price a "zero upfront" contract, meaning that equating the discounted expected payments of the protection buyer and seller gives

   $$\text{(buyer)} \qquad s^\star \sum_{t=1}^{5}(1-P(t))D(t) = \sum_{t=1}^{5}(P(t)-P(t-1))(1-R)(1+c)D(t) \qquad \text{(seller)} \quad (1)$$

   Using the constant hazard rate model, we plug in $P(t) = 1 - \exp[-\lambda t]$, and solve (1) numerically for λ given a quoted spread $s^\star$.

   The new (post-Big Bang) contract pays a standard spread of s, along with an upfront payment U. Equating the buyer and seller payments gives:

   $$\text{(buyer)} \qquad U + s\sum_{t=1}^{5}(1-P(t))D(t) = \sum_{t=1}^{5}(P(t)-P(t-1))(1-R)(1+c)D(t) \qquad \text{(seller)} \quad (2)$$

where the $P(t)$ are determined by the value of λ obtained above.

Subtracting (1) from (2) gives

$$U + (s - s^\star) \sum_{t=1}^{5} (1 - P(t)) D(t) = 0, \tag{3}$$

which gives the relationship between upfront and fair spread.

The numerical procedure is illustrated in the spreadsheet. For example, for a CDS referencing a bond with 4% coupon and with 40% assumed recovery, a fair spread of 120bp implies $\lambda = 0.0190$; using a standard spread of 100bp would require the protection buyer to make an upfront payment of 0.92%.

3. The key observation here is that a default event on Bank of America would apply to all classes of debt, and therefore the default probability implied by the senior CDS should be what we use to price the subordinated CDS. The important difference between the two CDS is the expectation on recovery. Let s_k denote the quoted fair spread for the CDS on seniority class k ($k =$ Sen or Sub); let c_k denote the coupon on the bond of seniority class k referenced by the CDS; and let R_k be the expected recovery for the bond of seniority class k. Then the pricing function gives

$$s_k \sum_{t=1}^{5} (1 - P(t)) D(t) = \sum_{t=1}^{5} (P(t) - P(t-1))(1 - R_k)(1 + c_k) D(t), \; k = \text{Sen, Sub}. \tag{4}$$

Dividing this equation for the two seniority classes gives

$$\frac{s_{\text{Sen}}}{s_{\text{Sub}}} = \frac{1 - R_{\text{Sen}}}{1 - R_{\text{Sub}}} \times \frac{1 + c_{\text{Sen}}}{1 + c_{\text{Sub}}}. \tag{5}$$

Assuming for simplicity that the coupons of the two bonds are equal, the spread scales with the ratio of the loss rates on the two classes of debt.

The subordinated debt should not recover more than the senior debt, so we have a lower bound for the Subordinated CDS at 60bp.

Referring to the Altman and Kishore data from the slides, we see that a high recovery for Senior Unsecured (that is, one standard deviation above the mean) is 74.4%, and a low recovery for Subordinated (one standard deviation below the mean) is 8.9%. This gives a ratio of loss rates of 0.28, implying a spread for the subordinated CDS of 60bp/0.28=214bp.

4. The model fitting and CAP calculation is presented in the spreadsheet for the two variable case. For the variable pair (X_3, X_4) – that is, EBIT/TA and ME/BL – we obtain an accuracy ratio of 90.1%. We observe that the testing data is perfectly partitioned by the variable X_2 (RE/TA). Interestingly, the model using all variables does not perform perfectly on the out-of-sample data, and the model using only X_2 and X_3 is more robust. The accuracy ratios for all of the model specifications are listed on the far right of the spreadsheet.

5. In the spreadsheet, I have examined the transformed variable $X_3^\star = -\log[100 - X_3]$. The fitting with $(X_3^\star, x_4)$ is also illustrated in the spreadsheet. The accuracy ratio for this pair is 98.3%. Results for transforming variables X_2 and X_3 are also listed. Both transformations improve the out-of-sample fit slightly.

6. The pricer is in the solutions spreadsheet. We can observe that P increases with T. For instance, using the historical volatility (32%), and increasing T to 10, gives $P = 8.31\%$. This is an obvious effect, since a larger T gives a greater likelihood that the asset value process goes below D. But we also observe that the annualized probability p increases with T as well. (Increasing T to 10 gives an annualized p of 0.86%. This is not obvious, nor is it necessarily desirable. In general, we consider firms with more stable (that is, longer maturity) sources of funding to be more stable: short maturity funding brings the risk of refinancing. In many cases, firms default because they are unable to roll over (that is, refinance) their short term debt. This would imply some dynamic where, other things equal, a default was more likely when T was smaller. This effect, however, is not a part of the Merton model.

7. First, assume a recovery rate (say $R = 0.4$) and solve the Hull-White CDS formula for λ, the hazard rate for each of the three firms. Given λ, we have the default probabilities $P(t)$ for the individual firms.

   Now, work in the general pricing framework discussed in the slides. Define the probability of no default occurring:
$$Q(t) = \mathcal{P}\{\text{No defaults occur by time } t\}. \tag{6}$$
   At each time $t = 1, \ldots, 5$, the protection buyer will pay s if the first default had not occurred by the end of the prior period $(t-1)$, and zero otherwise. So the expected value of the discounted payments by the buyer is
$$s \sum_{t=1}^{5} D(t)Q(t-1). \tag{7}$$
   At each time t, the protection seller pays an amount of $1-R$ if the first default occurs between $t-1$ and t, and pays zero otherwise. In other words, the seller makes a payment if there are no defaults by $t-1$ and at least one default by time t. The expected value of the discounted payments by the seller are thus:
$$(1-R)\sum_{t=1}^{5} D(t)(Q(t-1) - Q(t)). \tag{8}$$
   To solve for the fair spread, we find the value of s that equates the buyer and seller payments:
$$s^\star = \frac{(1-R)\sum_{t=1}^{5} D(t)(Q(t-1) - Q(t))}{\sum_{t=1}^{5} D(t)Q(t-1)}. \tag{9}$$
   To price the FTD in specific cases, it suffices to compute the function $Q(t)$.

   For the case where $\rho = 0$, the default events are independent. This implies that
$$Q(t) = (1 - P(t))^3. \tag{10}$$
   We may plug this in to (9) to obtain the fair spread.

   For the case where $\rho = 1$, the three obligors essentially act as one: either all default or none default. This implies
$$Q(t) = 1 - P(t). \tag{11}$$

Again, we may plug this in to (9) to obtain the fair spread.

For the general case, we need to apply the conditioning argument and integrate over z. Given $Z = z$, compute the conditional default probability

$$p(t,z) \equiv \mathcal{P}\{\text{Obligor defaults by } t | Z = z\} = \Phi\left(\frac{\Phi^{-1}(P(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right). \tag{12}$$

Define $Q(t,z)$ to be the conditional probability of no defaults by t, given $Z = z$. Using conditional independence, we have

$$Q(t,z) = (1 - p(t,z))^3 \tag{13}$$

It remains to integrate over z:

$$Q(t) = \int dz\, \phi(z) Q(t,z), \tag{14}$$

where ϕ denotes the probability density function for the standard Gaussian distribution. This can be computed numerically.

We can observe that as ρ increases, Q increases as well, and the fair spread $s^\star$ decreases. If a protection buyer enters a contract at a fair spread based on current correlation levels, and then correlation increases, the protection buyer will then be paying more (the fair spread at which he entered the contract) than the market level (the new, lower fair spread based on the higher correlation). In other words, a higher correlation will produce a negative mark-to-market change for the protection buyer. Said differently, an increase in correlation will lower the probability that the protection seller will have to pay out, which clearly is a benefit to the seller. Thus, the protection buyer is "short correlation" and the protection seller is "long correlation".