Empirical models of corporate credit quality

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Agenda

- Some motivation
- Market-based credit indicators, derivatives pricing
- Fundamental credit models
- Structural (Merton) credit model
- Portfolio models
- Pricing structured credit

Why do we care?

- Lending decisions
- Pricing credit
- Ratings arbitrage
- Relative value
- Reserves policy
- Allocating economic capital
- Allocating regulatory capital

MARKET CREDIT INDICATORS

How does the market express the price of credit?

- In its traditional form, credit is traded as bonds.
- But bonds encompass interest rate risk as well as credit risk, so how to disentangle?
- What is a better value, a government bond paying 2% interest or a corporate bond paying 4%?

Indicators of priced credit risk

- Consider a fixed coupon bond with maturity *T*, paying annual coupon *c.*
- Yield-to-Maturity (YTM) is defined in the normal (implicit) way in order to recover the market price.

$$P = c\sum_{i=1}^{T} \frac{1}{(1+y)^i} + \frac{1}{(1+y)^T}$$

- Recall weaknesses of the YTM indicator.

Indicators of priced credit risk

- Yield-based indicators
  - Yield spread – the difference between the YTM for the credit risky bond and the YTM of the closest benchmark government bond
  - Interpolated spread – the difference between the YTM for the risky bond and the interpolated government YTM for the actual bond maturity

- These inherit the weaknesses of YTM, and do not generalize to more complex securities.

Indicators of priced credit risk

- Option-adjusted spread (OAS) or Z-spread
  - The constant spread required to add to the government discount curve in order to recover the risky bond price

$$P = c\sum_{i=1}^{T} \exp[-i(Z_i + s)] + \exp[-T(Z_T + s)]$$

- Accounts for the coupon effect, and generalizes to more complex securities.
- See O'Kane and Sen (2005) for further discussion.

Recall basic definitions

- Let *T* denote the (random) time to default of a specific obligor.
- Let *P* be the cumulative distribution function for *T*.
- Then we have
  - Default probability $\mathcal{P}\{T \leq t\} = P(t)$
  - Survival probability $\mathcal{P}\{T > t\} = 1 - P(t)$
  - Unconditional probability of default between *t* and *t+h*

$$\mathcal{P}\{T \in (t, t+h]\} = P(t+h) - P(t)$$

  - Conditional probability of a default between *t* and *t+h*, given survival until *t*

$$\mathcal{P}\{T \in (t, t+h] | T > t\} = \frac{P(t+h) - P(t)}{1 - P(t)}$$

Work with the simplest distribution

- Let the default time be exponentially distributed with parameter λ
- Then default probability is $P(t) = 1 - \exp[-\lambda t]$
- Survival probability is $1 - P(t) = \exp[-\lambda t]$
- Unconditional default probability in (t,t+h]

$$P(t+h) - P(t) = \exp[-\lambda t] - \exp[-\lambda(t+h)] = \exp[-\lambda t](1 - \exp[-\lambda h])$$

- Conditional default probability in (t,t+h] given survival to *t*

$$\frac{P(t+h) - P(t)}{1 - P(t)} = 1 - \exp[-\lambda h]$$

- Conditional default rate is constant. This is analogous to a constant instantaneous forward rate in interest rate models.

Implied default probabilities
Hull and White (2005)

- Default probabilities are governed λ, the hazard rate

$$\mathcal{P}\{\text{Default before}\, t\} = P(t) = 1 - \exp[-\lambda t]$$

  (This assumption can be relaxed.)

- Infer λ by assuming price is equal to its risk-free value less the expected loss due to default

$$\begin{aligned} P &= c\sum_{i=1}^{T} e^{-iZ_i} + e^{-TZ_T} \ldots \\ &\quad - \sum_{i=1}^{T} (P(i) - P(i-1))(1-R)(1+c)e^{-iZ_i} \end{aligned}$$

- Default probabilities have a concrete meaning and are “portable” across security types and markets.
- Recall we are working under the risk-neutral measure here, so PDs embed the risk and liquidity premia.

Credit Default Swaps (CDS)

- Bilateral contract in which
  - *Protection buyer* pays an upfront premium (maybe), and then pays a periodic running premium until maturity of the contract
  - *Protection seller* pays out the default loss on a set of reference securities in the event of a credit event
- Credit event definition can be controversial, but think of it here as default, bankruptcy or major restructuring.
- Default loss is determined through an auction of the eligible securities one month after the credit event. Assume here that payment occurs on next premium date, and is simply $(1-R)(1+c)$

CDS Hidden Risks

- *Legal risk*. The protection buyer is at risk that an event that should be considered a default is not declared to be one. Consider the case of Greece in 2011/2012.
- *Counterparty risk*. The protection buyer (and to less of a degree seller) is exposed to the risk that the counterparty in the transaction fails to meet their future obligations. Consider the case of AIG in 2008.
- We will not consider these risks in the pricing to follow.

CDS pricing

- Assume annual premium of *s*, maturity *T.*
- Present value of protection buyer payments

$$\text{Upfront} + s\sum_{i=1}^{T}(1-P(i))e^{-iZ_i}$$

- Present value of protection seller payments

$$\sum_{i=1}^{T}(P(i)-P(i-1))(1-R)(1+c)e^{-iZ_i}$$

- Price calibration entails finding λ such that the two sides of the contract are of equal value.

Credit Indices (CDX, iTraxx)

- The majority of trading volume in credit derivatives is in index contracts
- A contract comprises standardized
  - Maturity date,
  - Fixed premium,
  - Basket of credits
- Protection seller compensates for default losses of underlying names, scaled by their weight in the basket
- Protection buyer pays a periodic premium (spread) on the remaining notional amount being protected
- Quotations are based on “fair spread” … this is converted to an upfront payment

A few comments about recovery

- Recovery rate is typically defined as the percentage of a *recovery claim* that an creditor receives after a default.
- Typically, this is reported as a non-discounted figure, not reflecting the timing risk.
- Bankruptcy proceedings can take years to complete. Recovery that comes out of the end of these proceedings is referred to as *ultimate recovery*.
- Securities continue to trade between the default date and the recovery date. These form part of the market for *distressed securities*.
- CDS are typically settled one month after the credit event, before the ultimate recovery is known. An auction is held for the defaulted securities in order to establish the recovery rate to be used for settling the CDS contracts.

Recovery data – recent CDS auctions

- Delta Airlines (2005), 18
- Delphi (2005), 63
- Lehman Bros (2008), 8.6
- Washington Mutl (2008), 57
- Ecuador (2009), 31
- General Motors (2009), 13
- General Motors loans (2009), 98
- CIT Group (2009), 68
- Japan Airlines (2010), 20
- AMBAC Financial (2010), 9.5
- Dynegy (2011), 71
- AMR (2011), 24
- Eastman Kodak (2012), 24
- Hellenic Republic (2012), 22

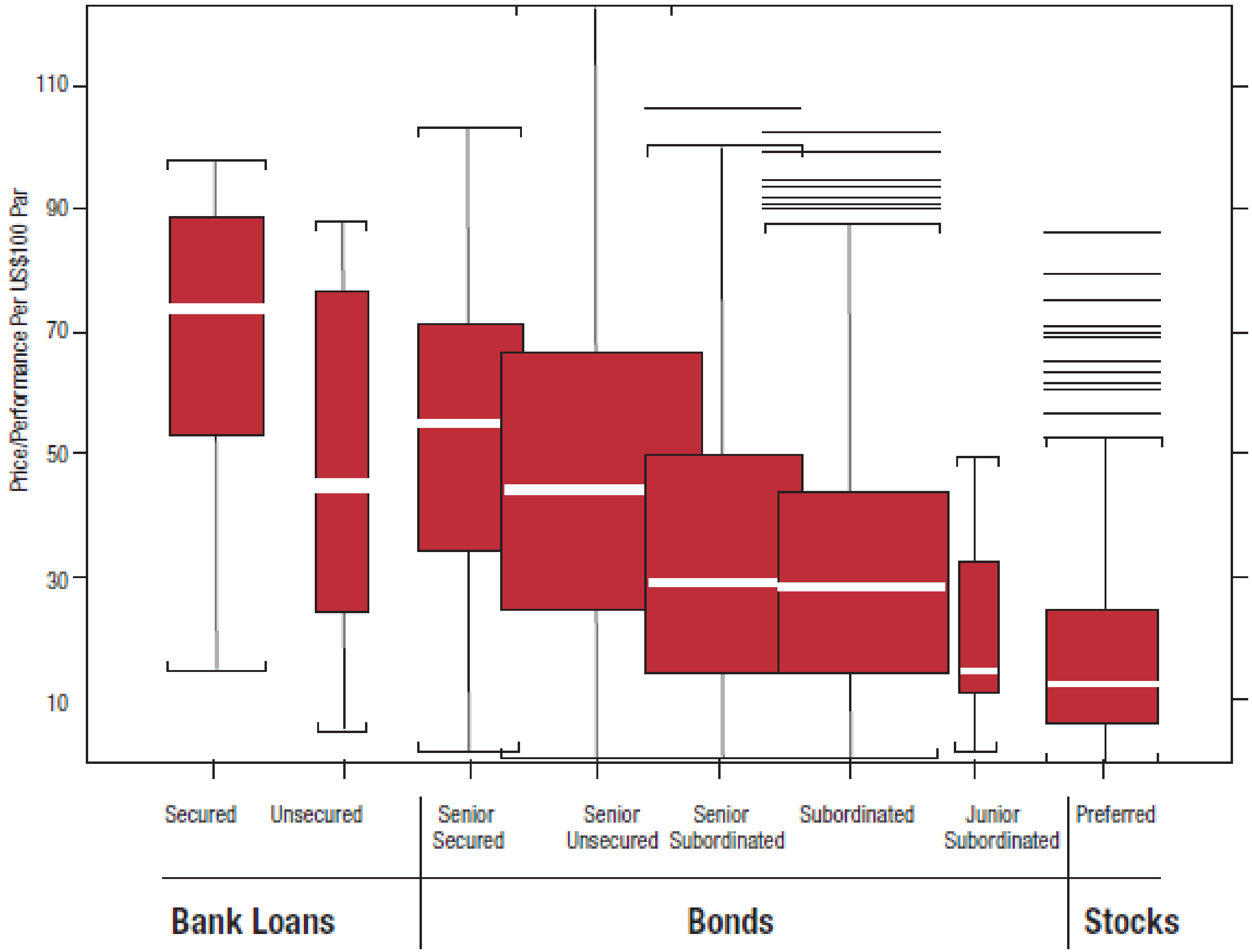
Source: www.creditfixings.com

Recovery data

- Historical data on ultimate recovery
- Altman and Kishore (1996), based on defaulted bonds from 1970-1995:

	Average (%)	Std. Dev. (%)
Senior Secured	57.9	23.0
Senior Unsecured	47.7	26.7
Senior Subordinated	34.4	25.1
Subordinated	31.3	22.4

Gupton and Stein (2002), data from 1981-2000



Altman and Kalotay (2010), data from 1987-2006

	Loans	*Bonds*	*Junior*	*Senior Sec*	*Sub*
Mean	0.75	0.52	0.16	0.59	0.27
Median	1.00	0.49	0.04	0.57	0.14
Std	0.33	0.39	0.26	0.34	0.34
IQR	0.53	0.85	0.22	0.79	0.42
10%	0.20	0.01	0.00	0.19	0.00
90%	1.00	1.00	0.52	1.00	0.98
N	637	2855	52	441	328

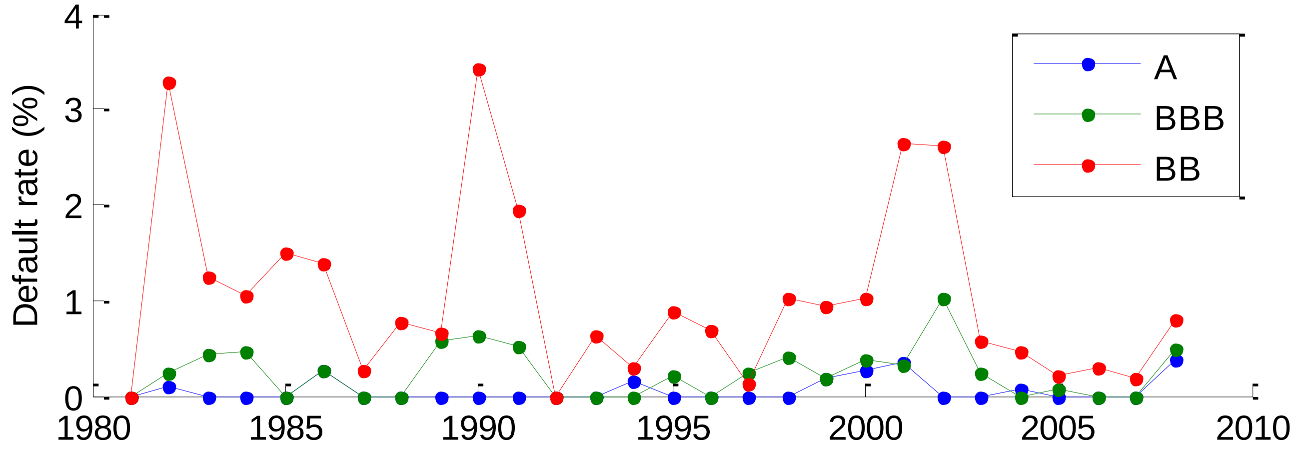
- Also investigate the effects of industry, macroeconomy, credit cycle, economic cushion …

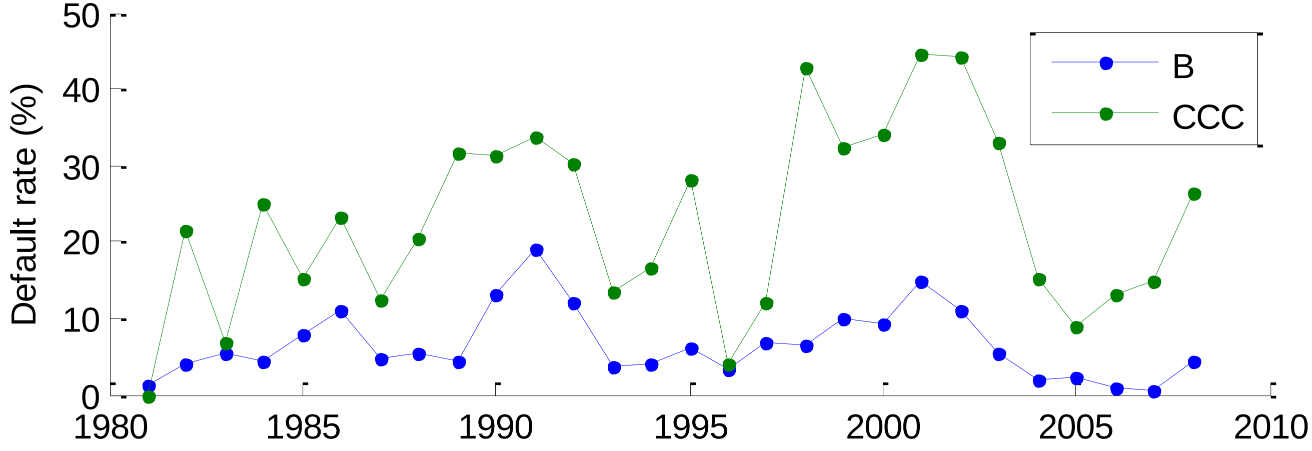
RATING MODELS

Standard & Poor’s criteria
Ratings goals (from company website)

- We view likelihood of default as the single most important dimension of creditworthiness.
- The key objective of Standard & Poor's ratings is rank ordering the relative creditworthiness of issuers and obligations.
- When our ratings perform as intended, securities with higher ratings should display lower observed default frequencies than securities with lower ratings during a given test period.
- In an indirect way, our consideration of absolute default likelihood can be viewed as associating "stress tests" or "scenarios" of varying severity with the different rating categories. We do not expect to observe constant default frequencies over time; we expect observed default frequencies for all rating categories to rise and fall with changes in economic conditions.
- Although we strive for comparability in our ratings, we expect to observe less consistency in rank ordering of observed default frequencies among regions and market segments.
- Only over very long periods - covering multiple economic cycles -would we expect to be able to observe whether similarly rated credits from different market segments actually experience similar long-term default frequencies.

History of one-year default rates





Ratings transition matrices

- For many modeling purposes, the probabilities of rating migration are an important input.
- Transition probabilities for a given year:
  - Begin with a cohort of all issuers (or issues) in a single rating at the beginning of the year.
  - At year end, tabulate the proportion in each possible new rating, in default, and with rating withdrawn. Proportion can be dollar weighted or issuer weighted.
- To calculate transition probabilities, make an assumption about the nature of withdrawn ratings.
  - Good – withdrawals because debt matures, count WR as no rating change
  - Bad – withdrawal because no longer able to access capital markets, count WR as default
  - Indifferent – average of the two, normalize other transitions (condition on no WR)
- Averaging across years … how to weight?
  - Each year's transition experience equally
  - Dollar or issuer weighting … more weight on recent years as issuance has increased
- Plenty of other issues related to forecasting … TTC/PIT, etc.

Example transition matrix

- Standard & Poor's non-modified system, 1985-2010

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	91.42	7.92	0.51	0.09	0.06	0	0	0
AA	0.61	90.68	7.91	0.61	0.05	0.11	0.02	0.01
A	0.05	1.99	91.43	5.86	0.43	0.16	0.03	0.04
BBB	0.02	0.17	4.08	89.94	4.55	0.79	0.18	0.27
BB	0.04	0.05	0.27	5.79	83.61	8.06	0.99	1.20
B	0	0.06	0.22	0.35	6.21	82.49	4.76	5.91
CCC	0	0	0.32	0.48	1.45	12.63	54.71	30.41
Default	0	0	0	0	0	0	0	100.00

FUNDAMENTAL MODELS

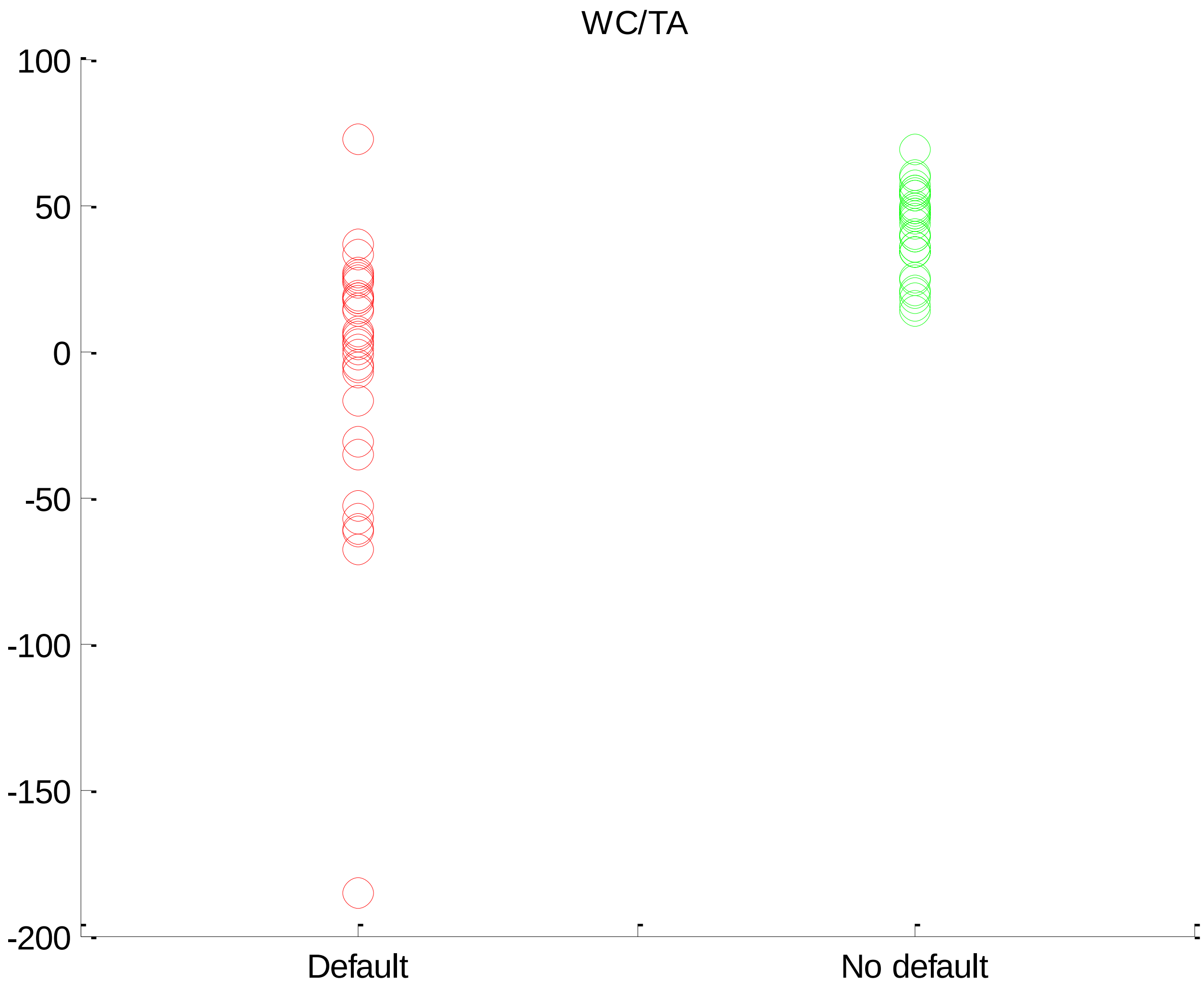
Z-Score

- Original study – Altman (1968)
  - Variable list … started with 22 variables grouped into liquidity, profitability, leverage, solvency, activity
  - 33 firms that filed for bankruptcy in 1946-65 … data from the year prior to filing (average lead time was about seven months)
  - 33 solvent firms selected randomly, with stratification to match industry and size distributions

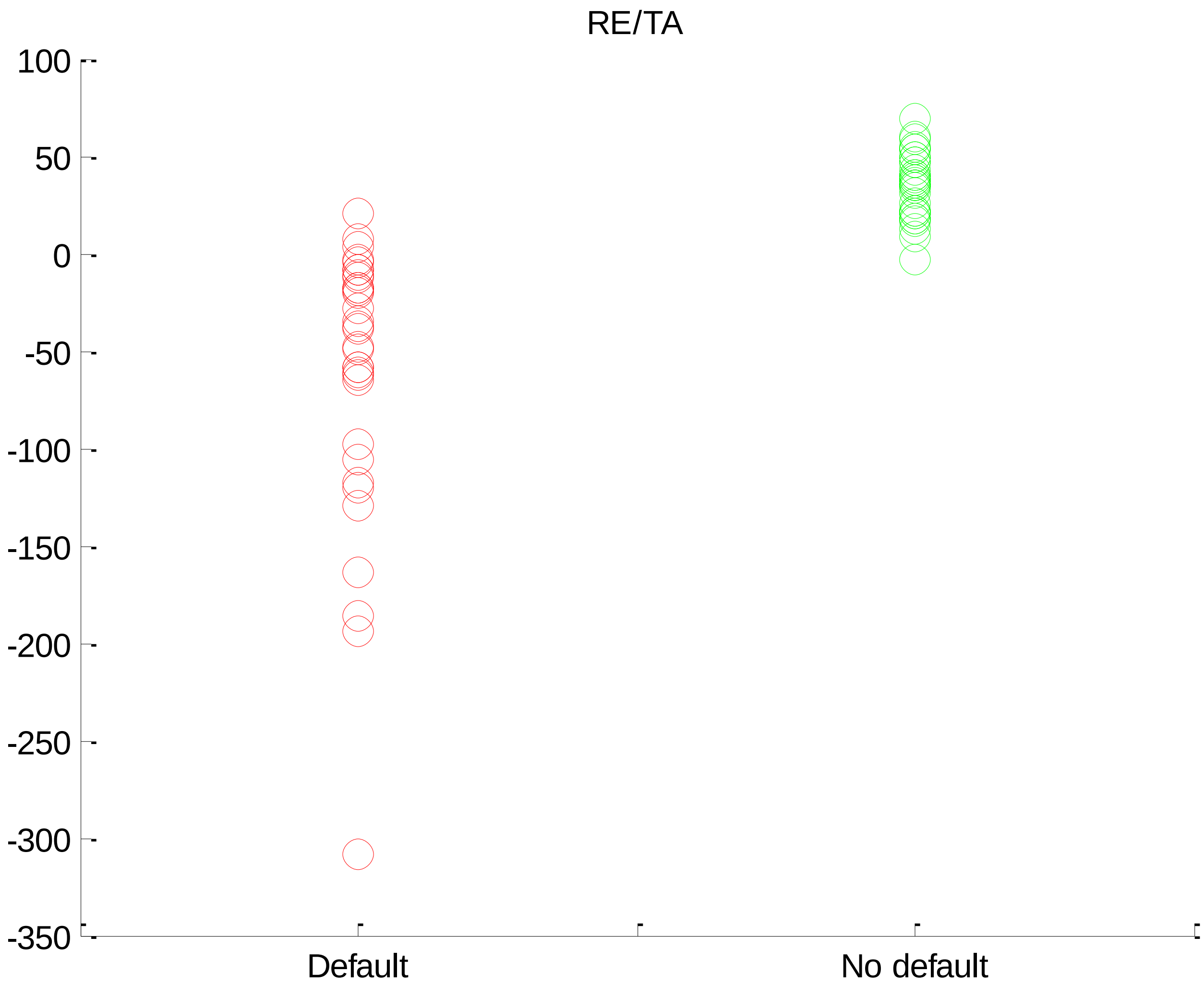
Z-Score … selected variables

- Working capital / total assets (WC/TA)
  - Measures net liquid assets
- Retained earnings / total assets (RE/TA)
  - Cumulative profitability
  - Implicitly penalizes young firms
  - Inversely related to leverage (high ratio means tendency to fund through earnings)
- EBIT / TA
  - Earnings power of firm's assets
- Market value of equity / Book value of total liabilities (ME/BL)
  - Market effect (already)
  - Leverage
- Sales / total assets (S/TA)
  - Poor power in univariate, but adds significantly in multivariate

Univariate discriminatory power



Univariate discriminatory power



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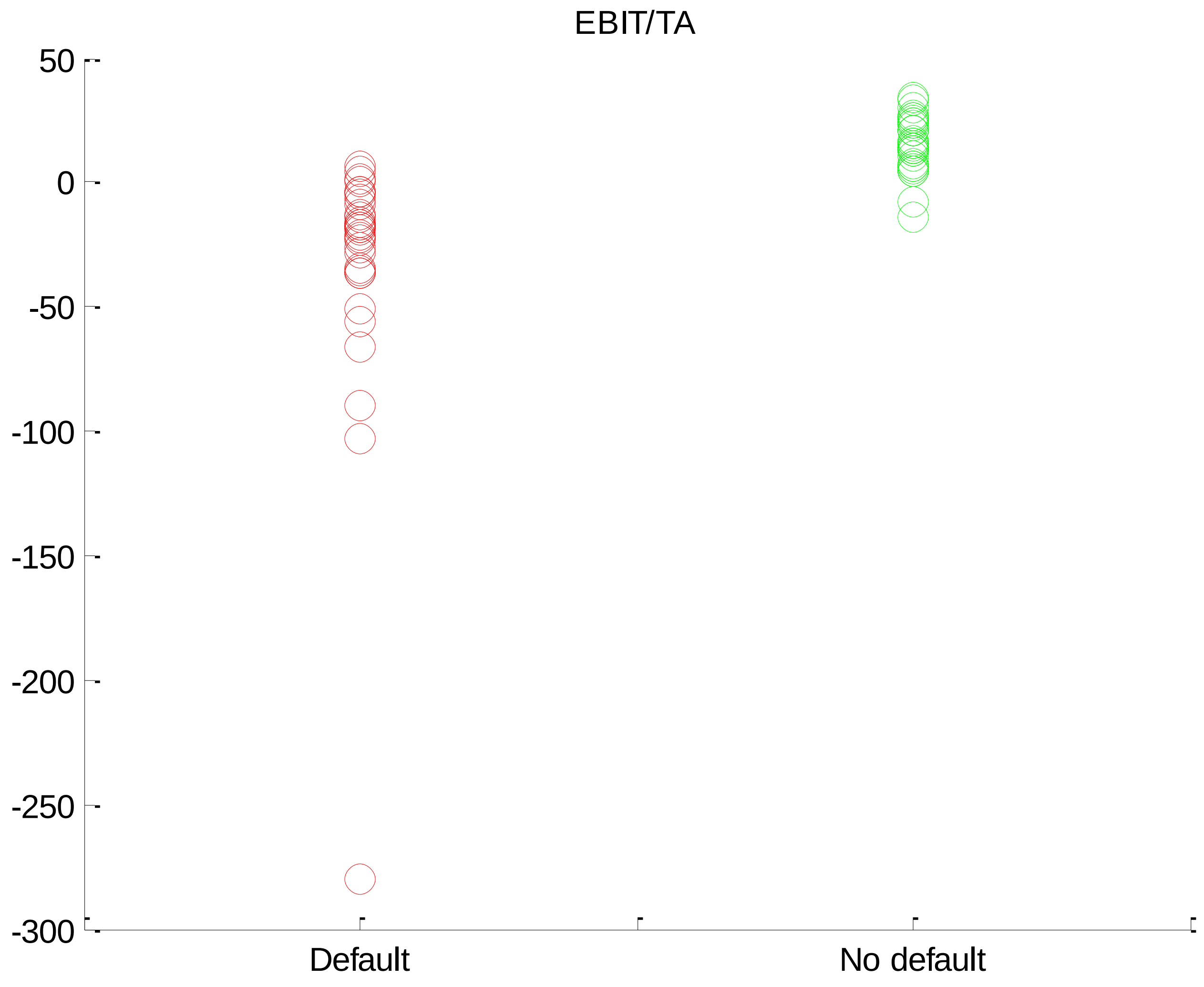
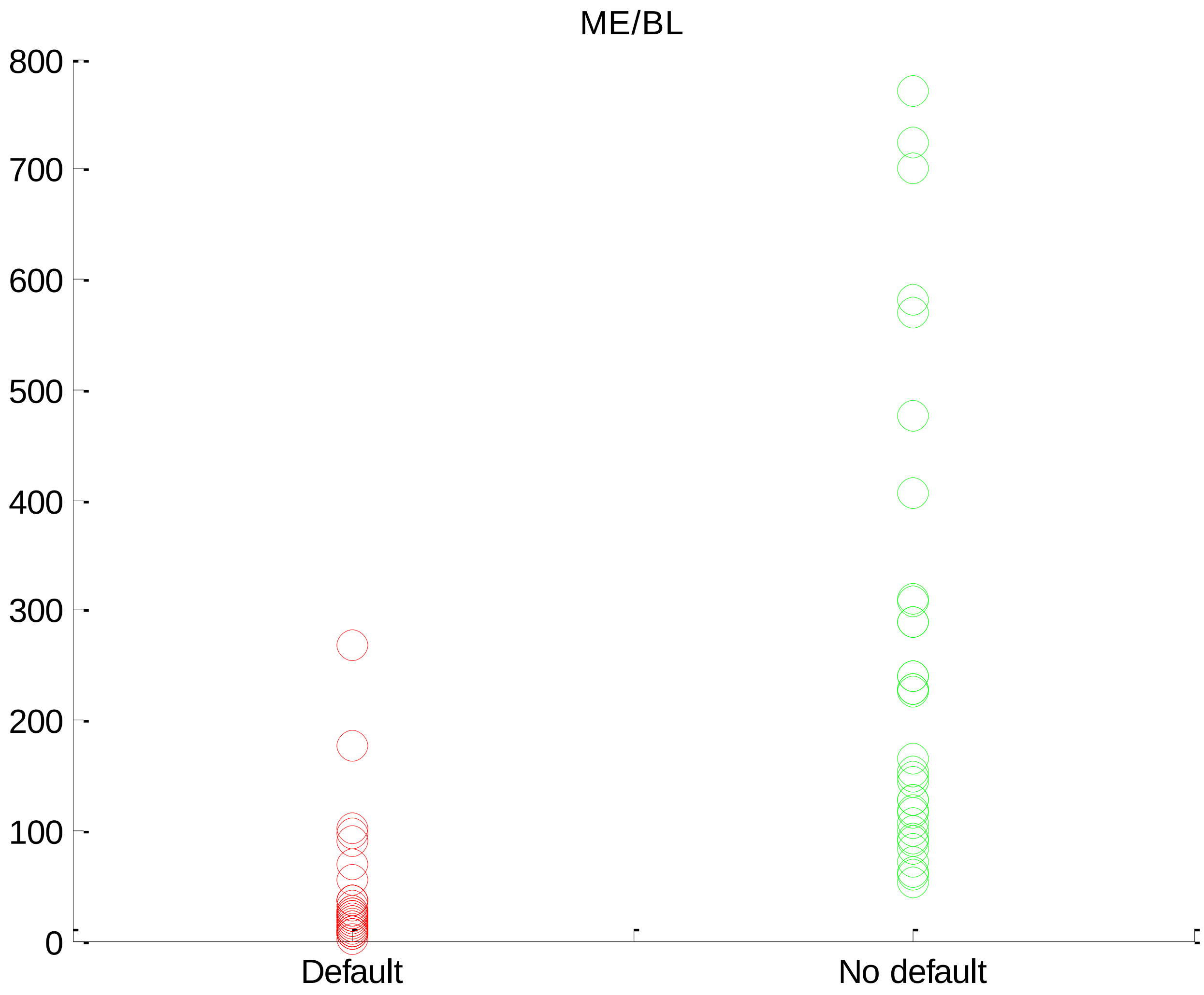


Univariate discriminatory power



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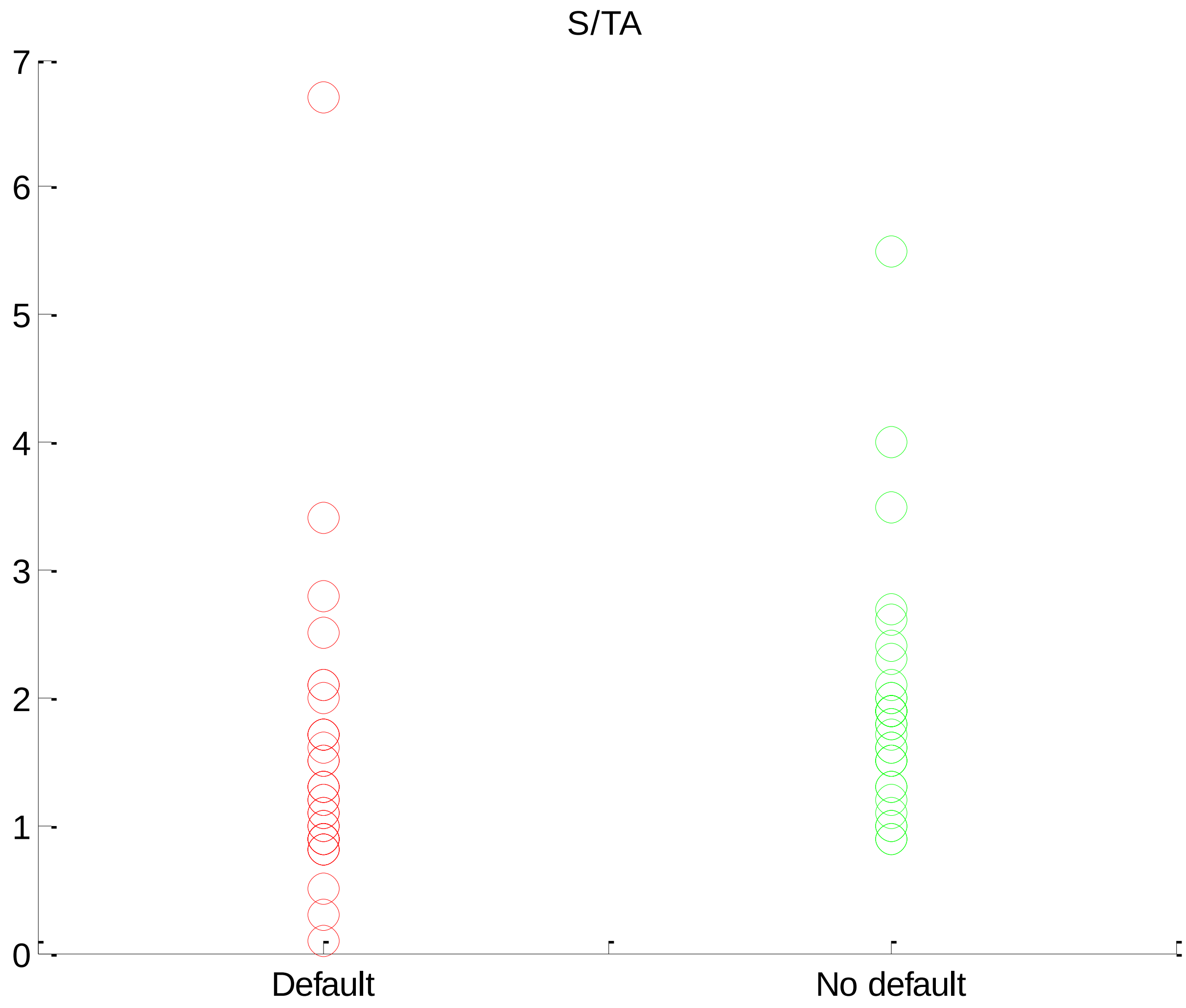
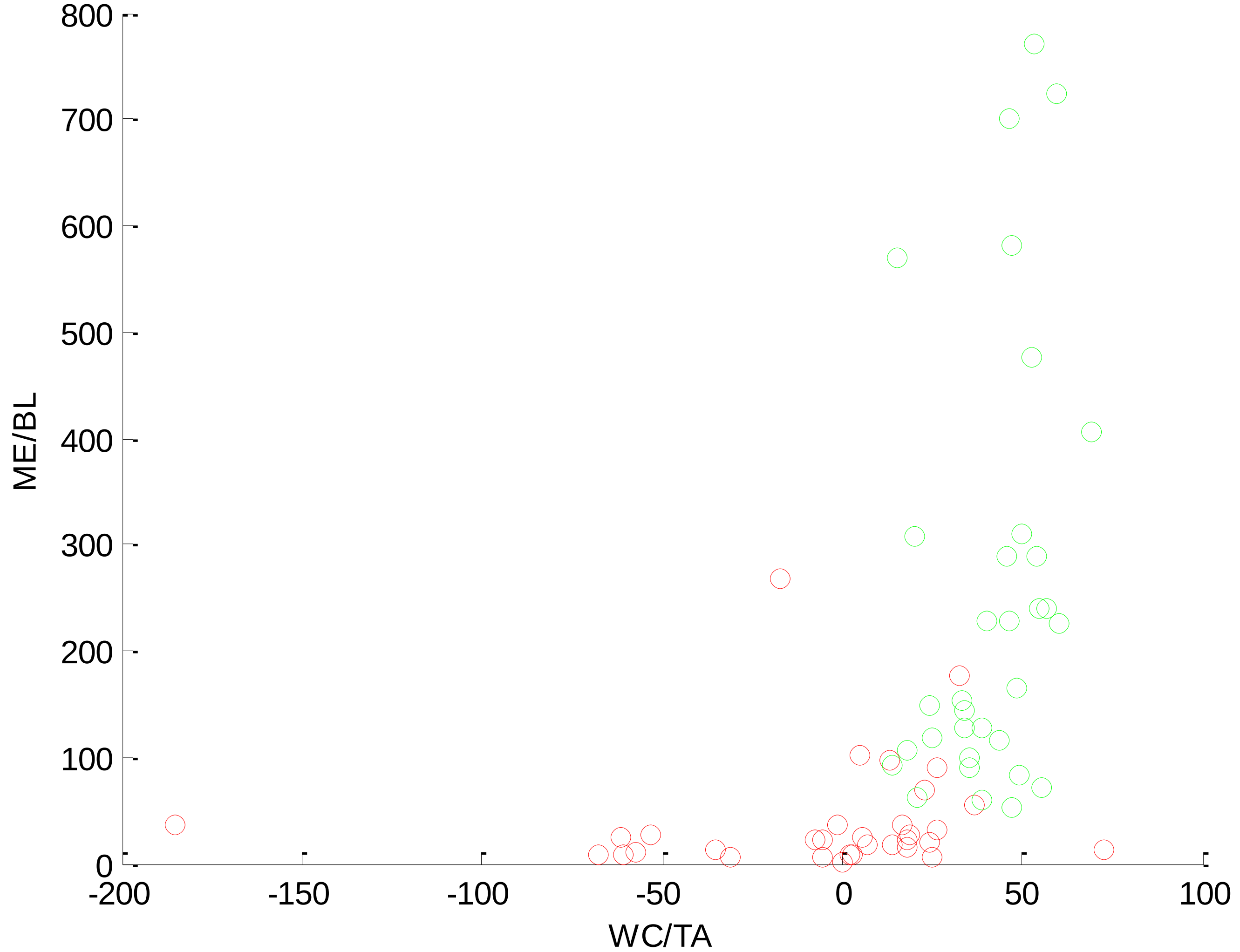
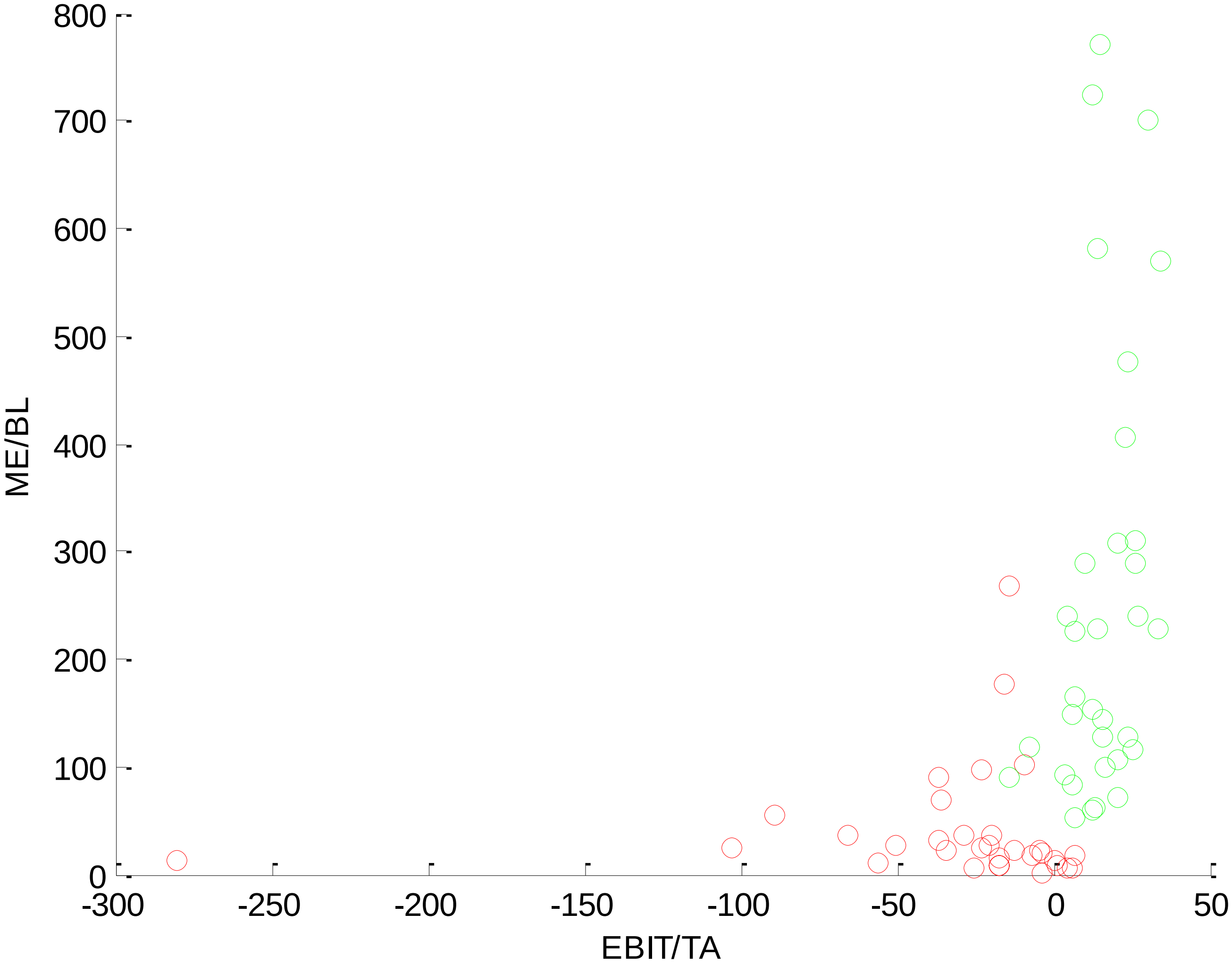


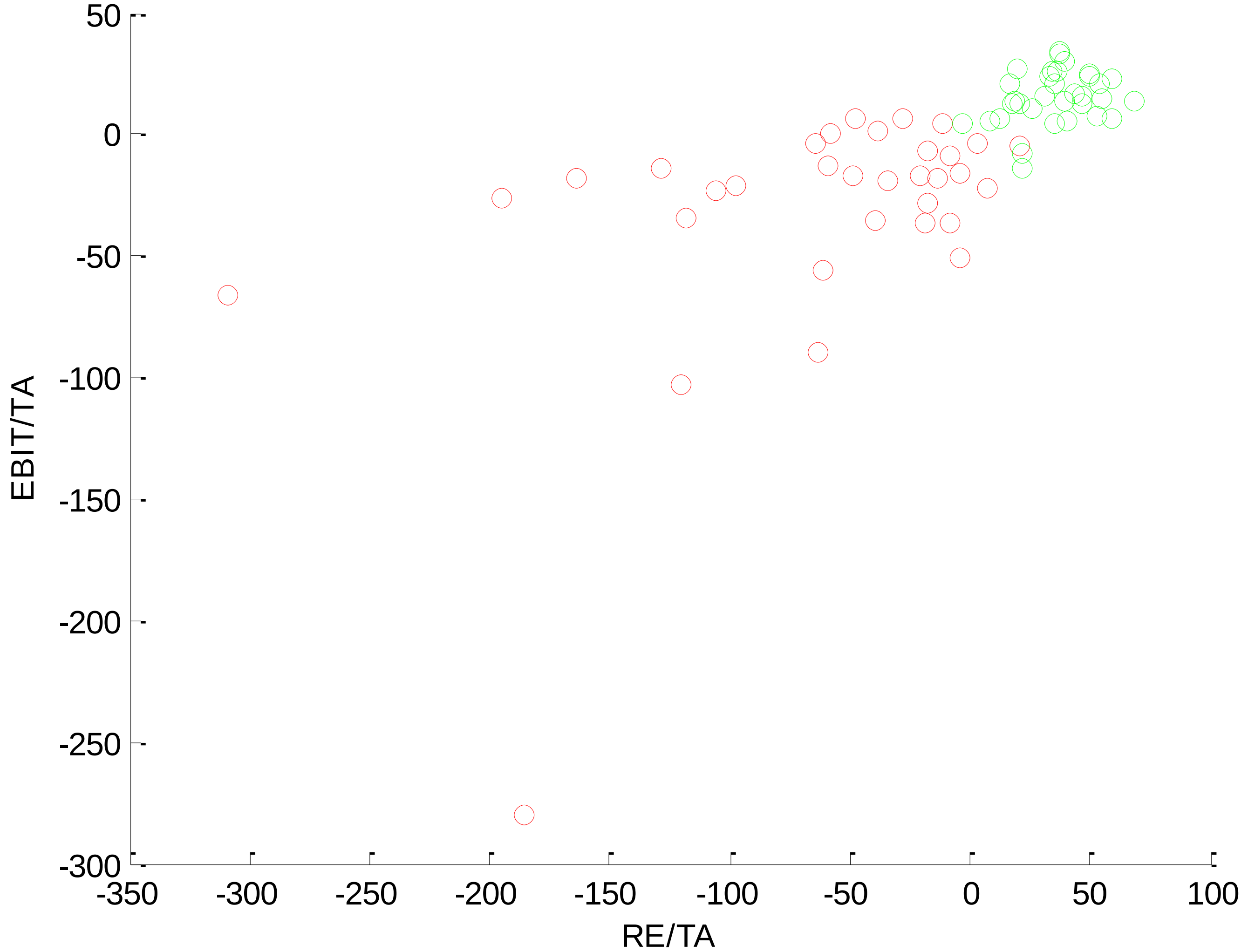
Bivariate discriminatory power



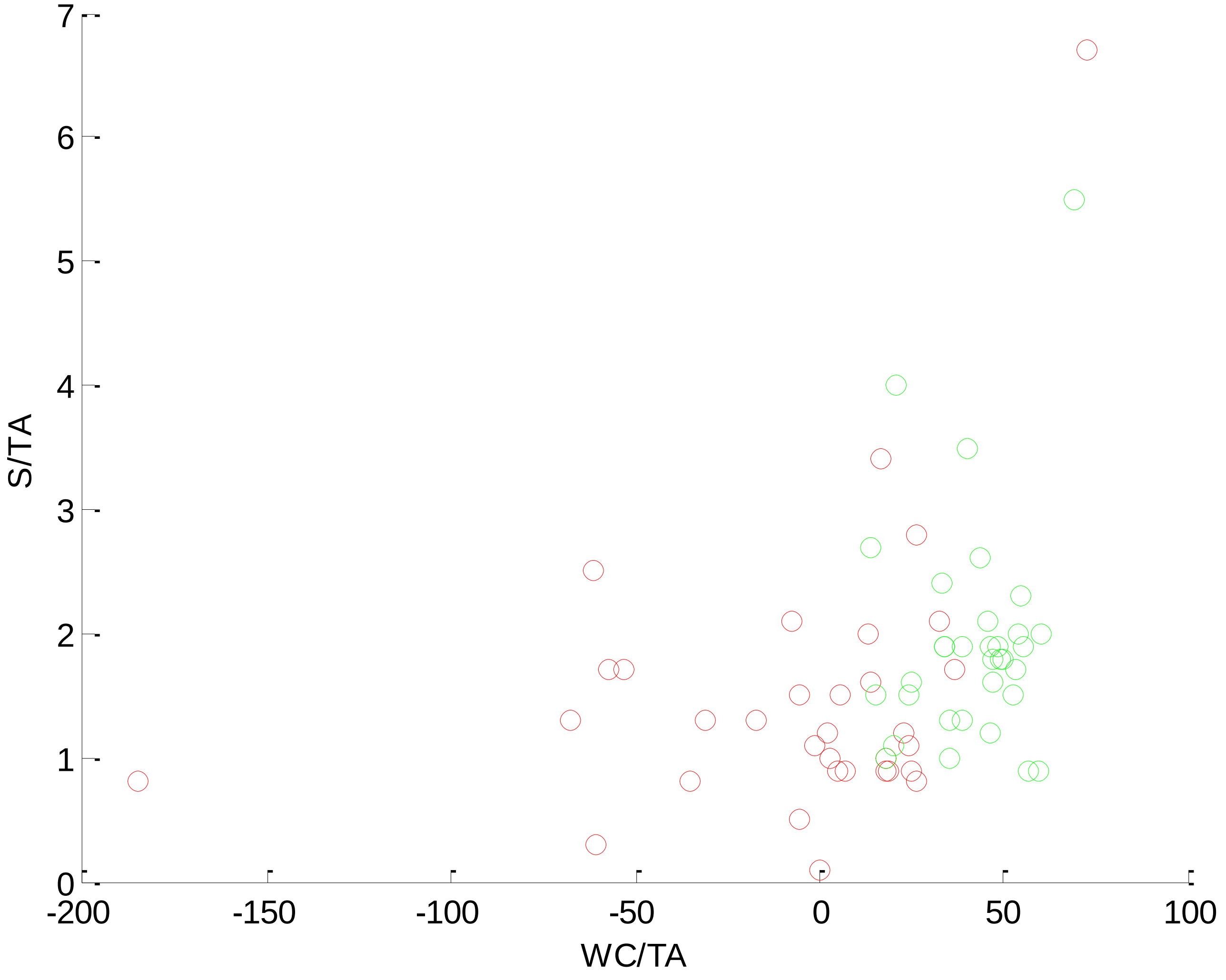
Bivariate discriminatory power



Bivariate discriminatory power



Bivariate discriminatory power



Linear discriminant analysis

- Vectors of descriptors $x_i, i = 1 \dots N$
- Assume the distribution of descriptors, conditional on knowing whether the firm defaulted/not, is Gaussian with mean $\mu_D \,/\, \mu_{ND}$ and covariance Σ
- It is natural to classify according to our estimate of the conditional probability of default given the descriptors -- $\mathcal{P}\{D|x\}$

Linear discriminant analysis

- Using Bayes' theorem

$$
\begin{aligned}
\mathcal{P}\{D|x\} &= \mathcal{P}\{x|D\}\frac{\mathcal{P}\{D\}}{\mathcal{P}\{x\}} \\
&= \phi(x-\mu_D,\Sigma)\frac{0.5}{0.5\phi(x-\mu_D,\Sigma)+0.5\phi(x-\mu_{ND},\Sigma)} \\
&= \frac{1}{1+\frac{\phi(x-\mu_{ND},\Sigma)}{\phi(x-\mu_D,\Sigma)}}
\end{aligned}
$$

- This is monotonic in the log likelihood ratio

$$
\log\frac{\phi(x-\mu_{ND},\Sigma)}{\phi(x-\mu_D,\Sigma)} = 2x'\Sigma^{-1}(\mu_D-\mu_{ND})+C
$$

Linear discriminant analysis

- So classification is performed according to the Z-Score

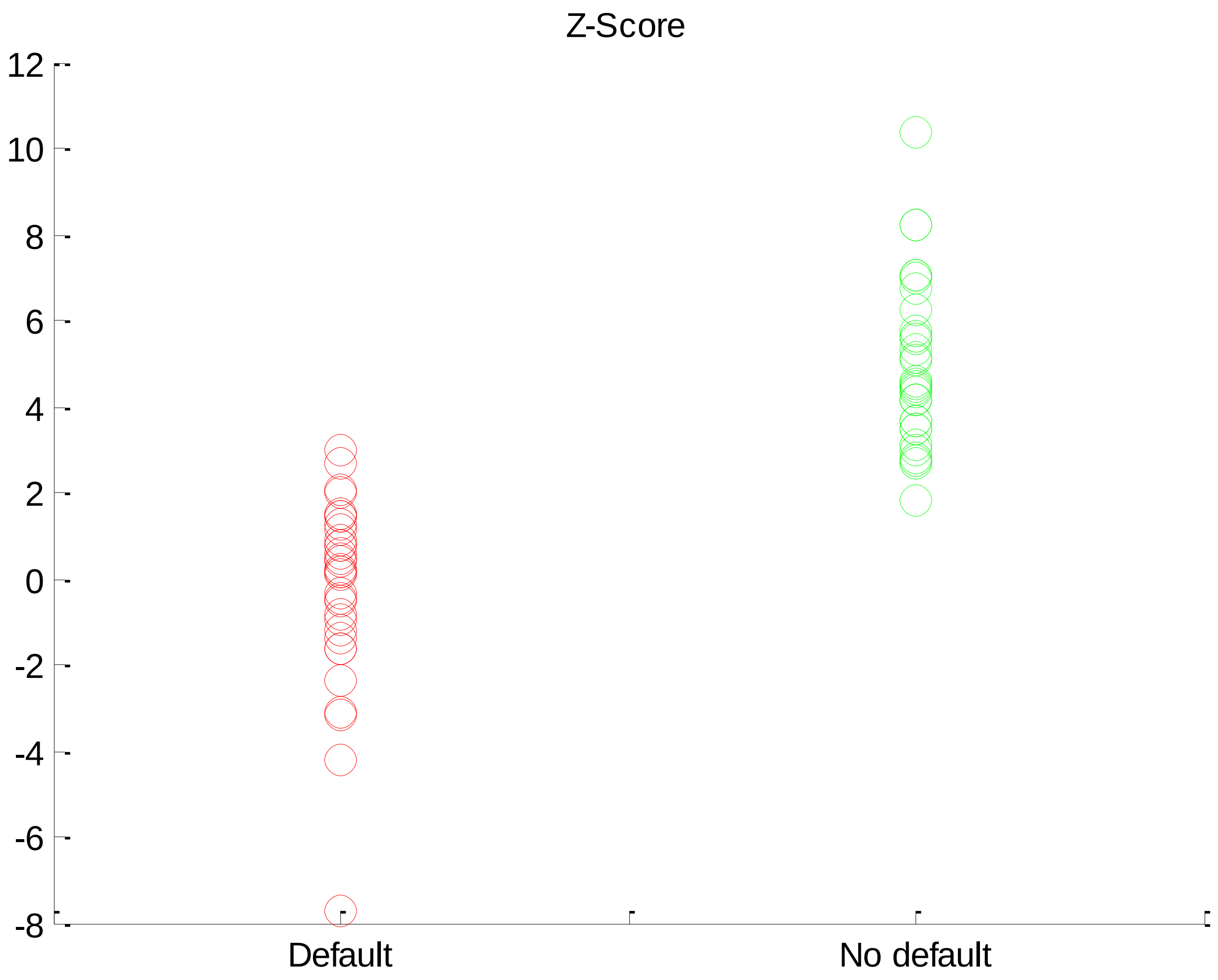
$$z = x'\Sigma^{-1}(\mu_D - \mu_{ND})$$

- This ranking can also be derived (Fisher (1936)) under weaker assumptions.
- Altman (1968) derives the Z-Score:

$$\begin{aligned} z &= 0.012(WC/TA) + 0.014(RE/TA) + 0.033(EBIT/TA) \\ &\quad +0.006(ME/BL) + 0.999(S/TA) \end{aligned}$$

- This is slightly different from the score given by the formula above.

Discriminatory power of Z-Score



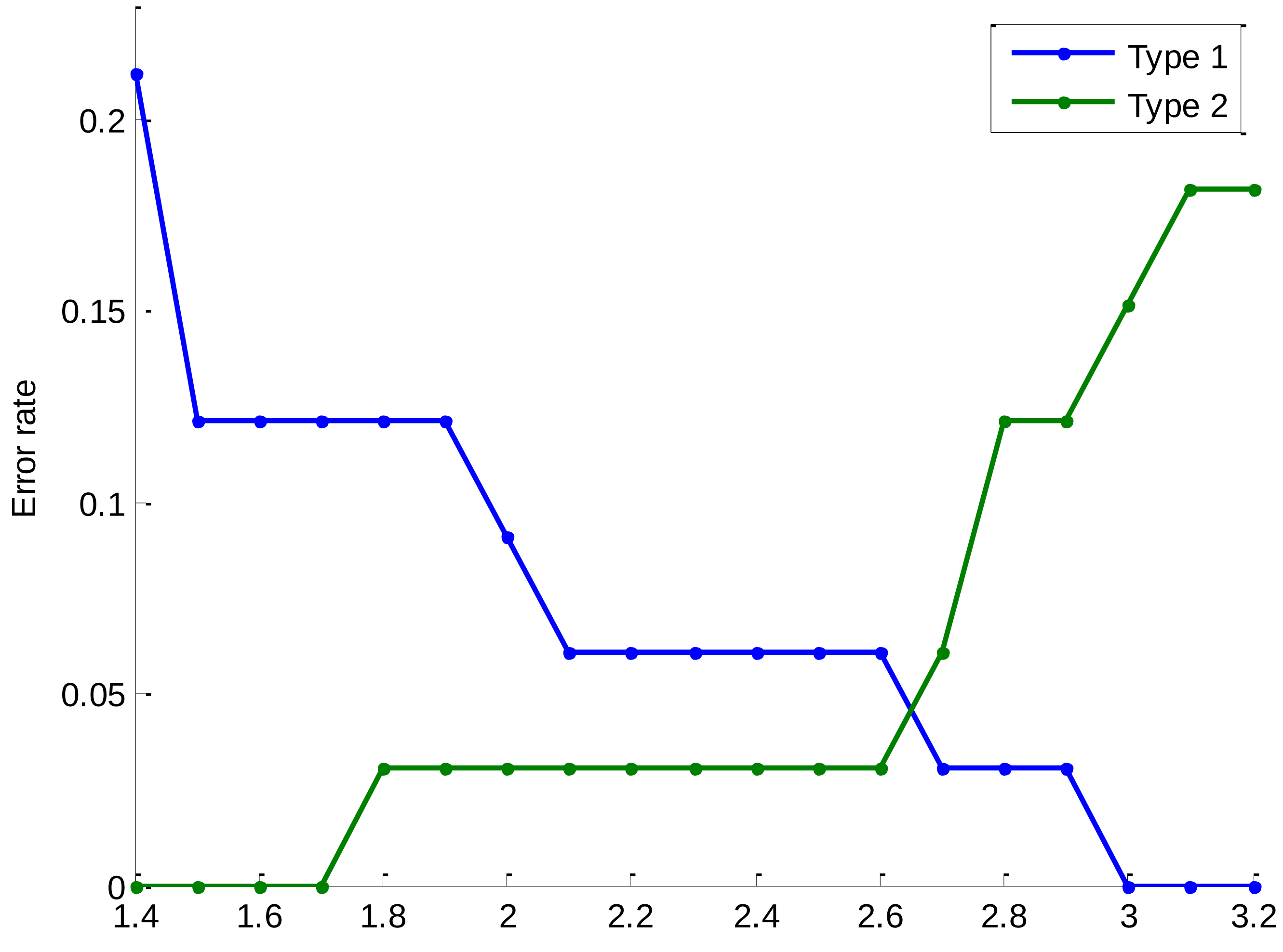
Linear discriminant analysis

- Note that this does not give us probabilities, just a ranking.
- How to choose the threshold?
- How to assess the model’s rankings?

Model assessment – error rates

- Any threshold we set is a tradeoff between two errors:
  - Type 1: characterizing an actual default as a non-default
  - Type 2: characterizing an actual non-default as a default

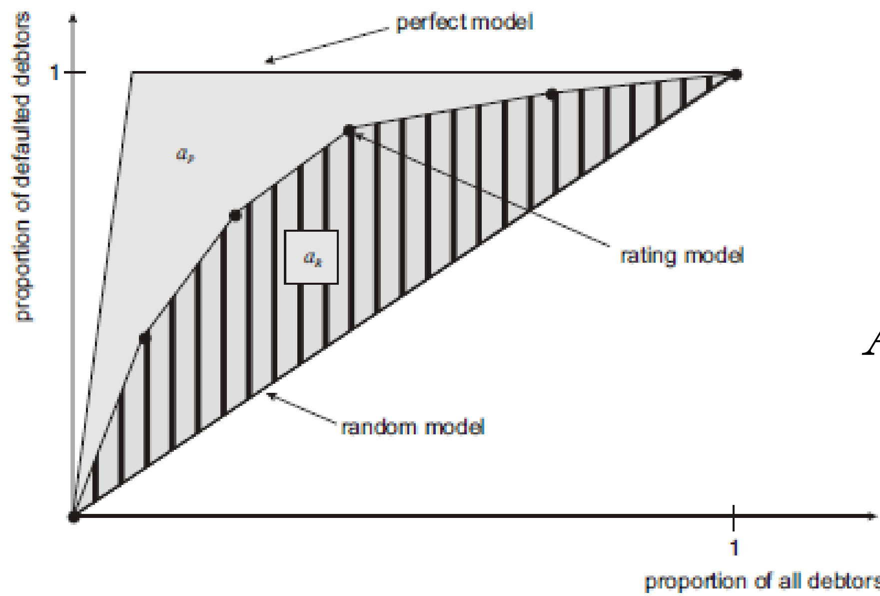
Model assessment – error rates



Model assessment – cumulative accuracy profile (CAP)

- For a given proportion of the total sample p, identify the firms with scores below the pth quantile.
- Calculate the proportion of defaulted firms $q(p)$ present in this subsample.
- The CAP is the graph of the pairs $\{p,q(p)\}$

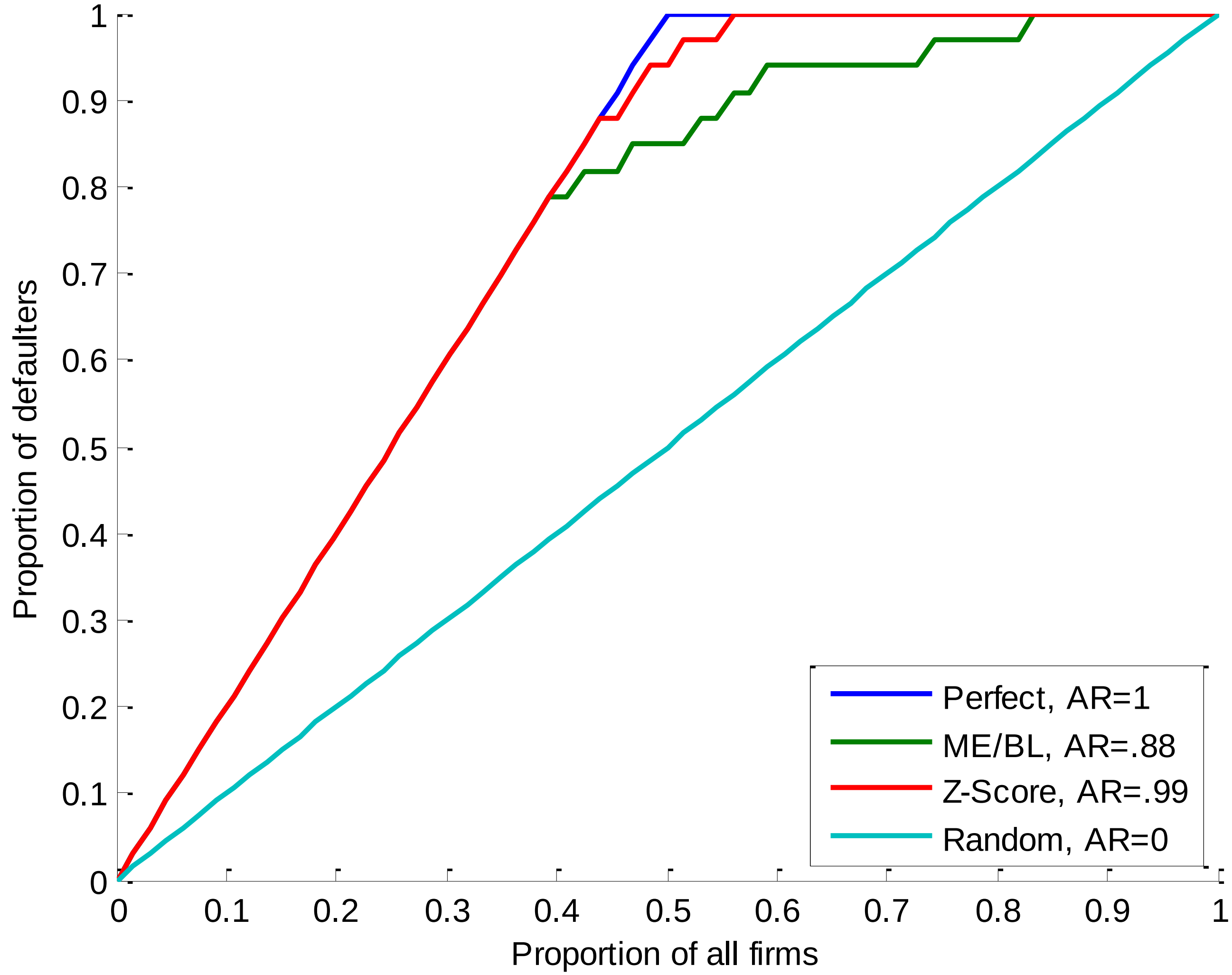
Model assessment – cumulative accuracy profile (CAP)



$$AR = \frac{a_R}{a_P}$$

- Engelmann et al (2002), Figure 1.
- The Accuracy Ratio (AR) is the ratio of the model performance to perfect performance, as measured by the area between the CAP curves.
- Curve is also known as Receiver Operating Characteristic (ROC).
- Statistics (area under curve) have a known standard error. See Fabozzi et al (2010) and references therein.

Model assessment – cumulative accuracy profile (CAP)



Some warnings

- All of these model assessments are *in sample*.
- All of the dependencies are linear.
- The data is over forty years old, and references manufacturing firms only. The results should not be extrapolated to the current day.
- The model only gives us relative scores, not default probabilities.

Extrapolate to the current day

- Consider MSCI, Inc. using data for year ending December 31, 2011

- From Income Statement (Yahoo Finance)
  – Income before tax (proxy for EBIT) -- $263.4M
  – Total revenue (proxy for Total Sales) -- $900.9M
- From Balance Sheet (Yahoo Finance)
  – Total current assets -- $677.9M
  – Total assets -- $3092.9M
  – Total current liabilities -- $452.8M
  – Retained earnings -- $363.5M
  – Total liabilities -- $1787.6M
- From Key Statistics (Yahoo Finance, 4 Dec 2012)
  – Market capitalization -- $3.55B

Extrapolate to the current day

- Now compute Z-Score inputs
  - WC/TA = 7.28
  - RE/TA = 11.75
  - EBIT/TA = 8.52
  - ME/TL = 199
  - TS/TA = 0.29
- Recall that first four items are expressed as percent, and the last as decimal.
- Z-Score = 2.02 (within the range of overlap between defaulters and non-defaulters)

Z-Metrics

- Data covers 1989-2008
- Over 260,000 observations (firm x time)
- 638 credit events (formal default or bankruptcy)
- Overall (unconditional) default rate is more representative than 50% (used implicitly in 1968)
- One- and five-year horizons

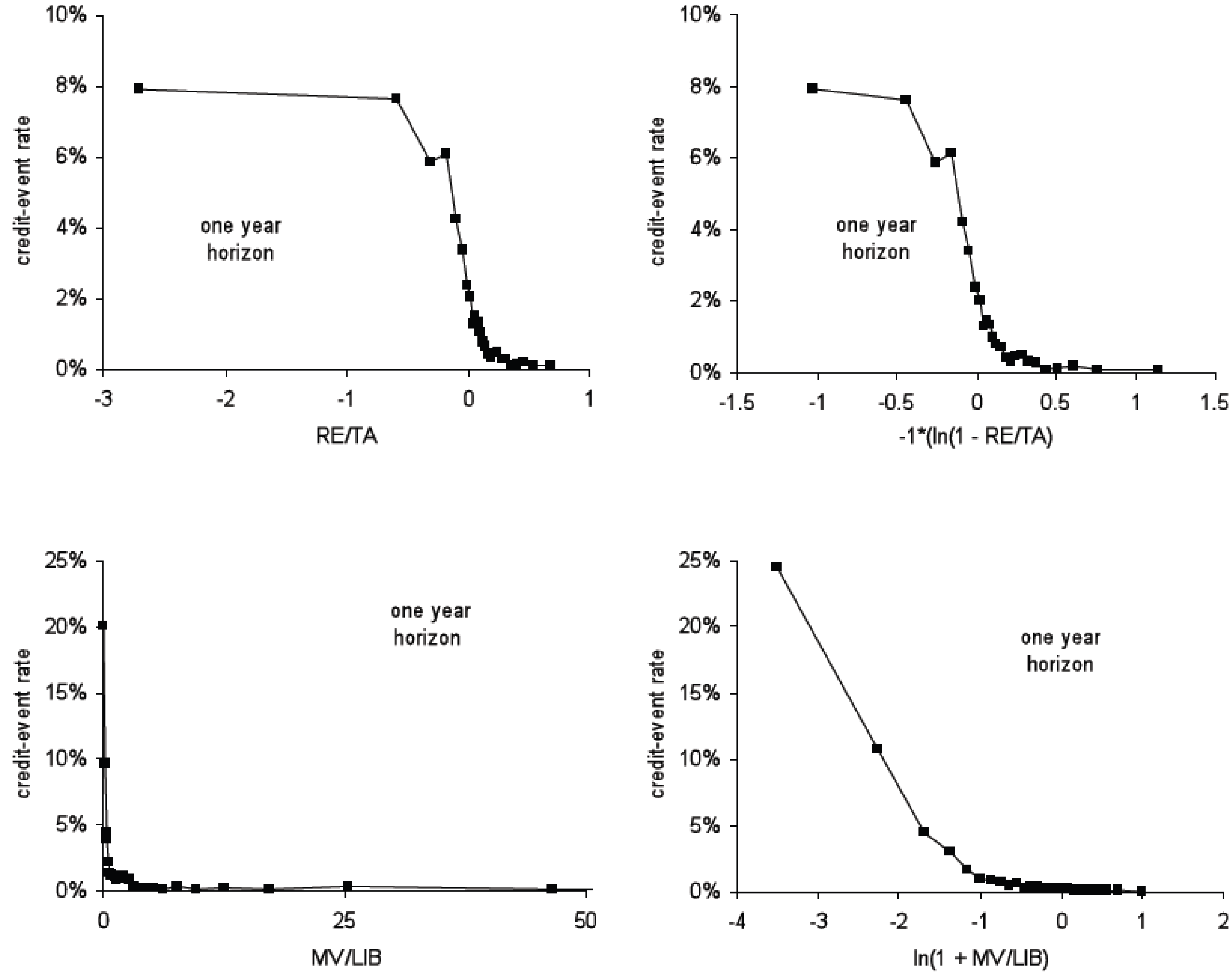
Z-Metrics – advances over 40 years

- More data – trends as well as levels
- Variable transformations – non-linear responses
- True point-in-time model – macroeconomic variables at time of observation to distinguish higher general levels of defaults
- Direct estimation of default probabilities through logistic regressions
- Stress tests

Z-Metrics variables

- Twelve fundamental variables
  - Financial statements
  - Market data
  - Trends
- Macroeconomic variables (one-year model)
  - Unemployment rate
  - Spread of high-yield bonds over 10yr Treasuries
- Transformations

Variable transformations
Z-Metrics White Paper, Figure 6



Model form

- Let *x* be a vector containing (transformed) fundamental and market-based variables for a specific firm at a specific time, as well as macroeconomic variables for this time.
- Use a logistic regression framework to model conditional default probabilities directly:

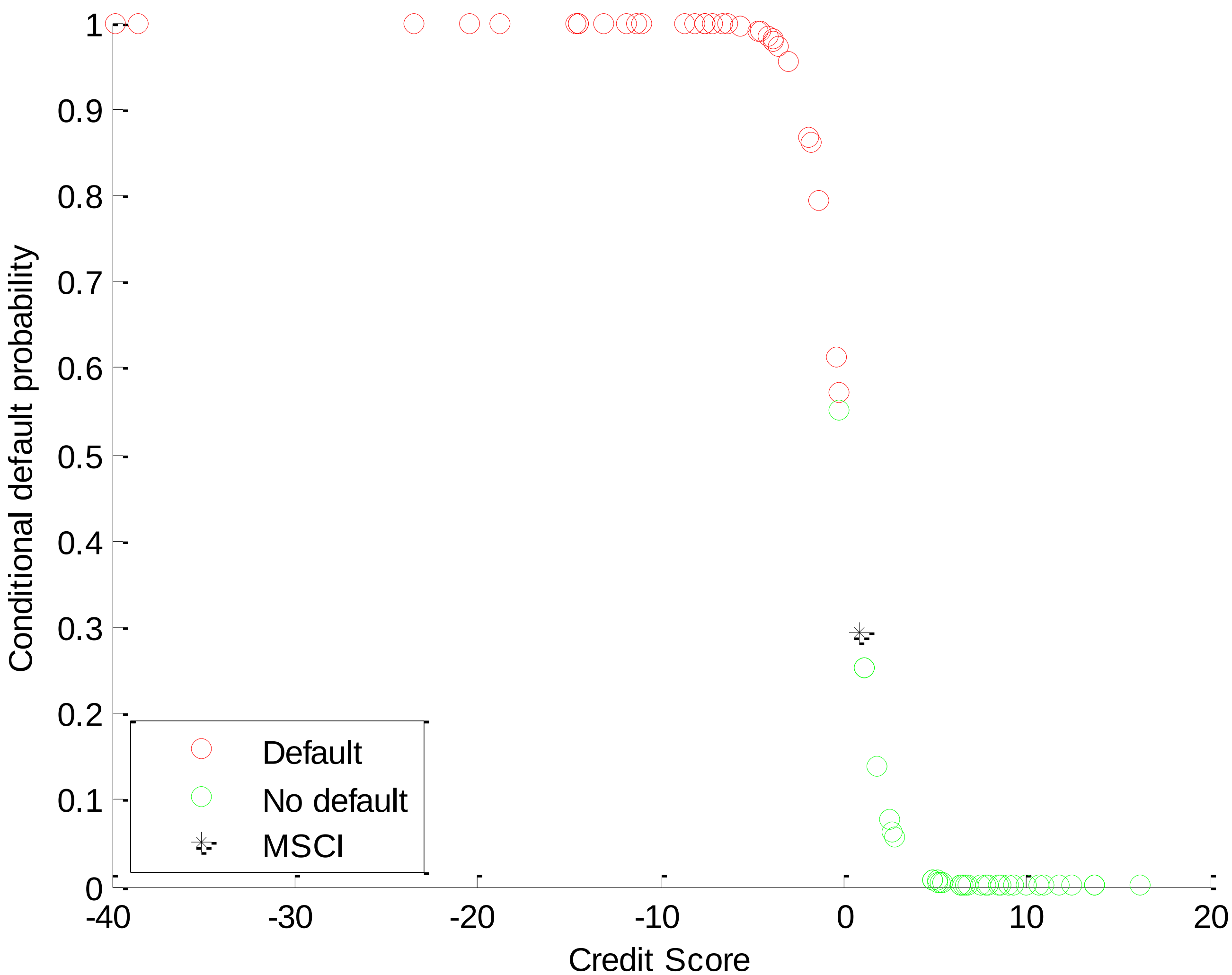
$$
\begin{aligned}
\mathrm{CS}(x) &= \alpha + \sum_i \beta_i x_i + \varepsilon \\
\mathcal{P}\{D|x\} &= \frac{1}{1 + \exp[\mathrm{CS}(x)]}
\end{aligned}
$$

Fitting the logistic regression

- Fit the coefficients α and β_i by Maximum Likelihood Estimation (see mnrfit in Matlab)
- Log Likelihood for overall estimation:

$$
\begin{aligned}
\log L &= \log\left[\prod_{j\in D}\frac{1}{1+\exp[\mathrm{CS}(x(j))]}\cdot\prod_{j\in ND}\left(1-\frac{1}{1+\exp[\mathrm{CS}(x(j))]}\right)\right] \\
&= \log\left[\prod_{j\in D}\frac{1}{1+\exp[\mathrm{CS}(x(j))]}\cdot\prod_{j\in ND}\frac{1}{1+\exp[-\mathrm{CS}(x(j))]}\right] \\
&= -\sum_{j\in D}\log[1+\exp[\mathrm{CS}(x(j))]]-\sum_{j\in ND}\log[1+\exp[-\mathrm{CS}(x(j))]]
\end{aligned}
$$

Fitting to 1968 data



Mapping default probabilities to ratings

Z-Metrics™ Ratings	Z-Metrics public - 1 Year				Z-Metrics public - 5 Years			
	one year PD		% representation		five year PD		% representation	
	min	max	1989/ 2008	2008	min	max	1989/ 2008	2008
ZA+	0.00%	0.02%	3.5%	2.1%	0.00%	0.75%	3.4%	2.4%
ZA	0.02%	0.04%	5.8%	4.6%	0.75%	1.25%	7.0%	5.4%
ZA-	0.04%	0.06%	7.6%	6.1%	1.25%	1.75%	7.6%	6.4%
ZB+	0.06%	0.09%	10.6%	10.0%	1.75%	2.50%	10.6%	9.9%
ZB	0.09%	0.14%	10.9%	11.2%	2.50%	3.50%	11.1%	11.3%
ZB-	0.14%	0.20%	8.8%	9.1%	3.50%	4.50%	8.1%	8.6%
ZC+	0.20%	0.30%	9.4%	10.8%	4.50%	6.00%	8.6%	9.7%
ZC	0.30%	0.50%	10.1%	10.4%	6.00%	9.00%	11.1%	12.1%
ZC-	0.50%	1.00%	10.6%	11.4%	9.00%	14.00%	10.0%	10.3%
ZD+	1%	2%	7.6%	8.2%	14%	20%	6.3%	6.8%
ZD	2%	4%	5.2%	5.8%	20%	30%	6.0%	6.6%
ZD-	4%	10%	4.5%	4.7%	30%	45%	4.5%	4.9%
ZF+	10%	25%	2.6%	2.6%	45%	65%	3.0%	3.2%
ZF	25%	50%	1.5%	1.6%	65%	80%	1.4%	1.6%
ZF-	50%	100%	1.2%	1.3%	80%	100%	1.0%	1.0%

Z-Metrics examples (as of 19 Nov 2010)

Issuer	One-year PD	One-year Z-rating	Five-year PD	Five-year Z-rating
MSCI INC-CL A	0.10%	ZB	4.49%	ZB-
MORNINGSTAR INC	0.04%	ZA	1.92%	ZB+
FACTSET RESEARCH SYSTEMS INC	0.01%	ZA+	0.32%	ZA+
APPLE INC	0.02%	ZA+	1.23%	ZA
AT&T INC	0.06%	ZB+	1.41%	ZA-
MICROSOFT CORP	0.04%	ZA-	1.06%	ZA
INTL BUSINESS MACHINES CORP	0.05%	ZA-	1.02%	ZA
JOHNSON & JOHNSON	0.02%	ZA+	0.50%	ZA+
LILLY (ELI) & CO	0.03%	ZA	0.63%	ZA+
MERCK & CO	0.03%	ZA	0.83%	ZA
MONSANTO CO	0.04%	ZA	1.13%	ZA
PFIZER INC	0.07%	ZB+	1.62%	ZA-
CROCS INC	0.04%	ZA	2.25%	ZB+
K-SWISS INC -CL A	0.28%	ZC+	7.53%	ZC
NIKE INC	0.01%	ZA+	0.49%	ZA+
SKECHERS U S A INC	0.15%	ZB-	4.11%	ZB-
GENERAL ELECTRIC CO	0.35%	ZC	6.26%	ZC
NUCOR CORP	0.09%	ZB	1.43%	ZA-
PG&E CORP	0.16%	ZB-	3.38%	ZB

STRUCTURAL MODELS

Merton (structural) model

- Consider model of Merton (1974), as described in Hull et al (2004).
- We model the (stylized) balance sheet of a firm:
  - Assets *A* evolve according to Geometric Brownian Motion with constant volatility σ_A
  - The firm issues one class of debt: a discount bond of size *D* maturing at time *T*
  - Equity receives no dividends
  - At *T*, assets are used to repay debt holders, with the residual going to equity holders

Valuing equity

- Equity holders' payout at *T*:

$$E_T = \max[A_T - D, 0]$$

- Under Black-Scholes-Merton, the current value of equity is:

$$\begin{aligned} E_0 &= A_0 N(d_1) - De^{-rt} N(d_2), \\ d_1 &= \frac{\log[A_0 e^{rt}/D]}{\sigma_A \sqrt{T}} + 0.5\sigma_A \sqrt{T}, \\ d_2 &= d_1 - \sigma_A \sqrt{T}. \end{aligned}$$

Valuing equity

- Define leverage ratio

$$L = De^{-rT}/A_0$$

- Then rewrite equity formula:

$$\begin{aligned} E_0 &= A_0[N(d_1) - LN(d_2)] \\ d_1 &= \frac{-\log[L]}{\sigma_A\sqrt{T}} + 0.5\sigma_A\sqrt{T}, \\ d_2 &= d_1 - \sigma_A\sqrt{T}. \end{aligned} \tag{1}$$

Calibrating

- Since equity value is a function of asset value, Ito's lemma gives instantaneous equity volatility:

$$
\begin{aligned}
dE_t &= \delta(A_t)dA_t + \frac{1}{2}\gamma(A_t)\sigma^2 A_t^2 dt \\
\sigma_E &= \mathrm{StDev}\frac{dE_t}{E_t} \\
E_0\sigma_E &= \delta(A_0)A_0\sigma_A \\
\sigma_E &= \frac{\sigma_A N(d_1)}{N(d_1) - LN(d_2)} \qquad (2)
\end{aligned}
$$

- Observe that equity is no longer a GBM, implying that the Black-Scholes framework no longer holds.
- To calibrate, we observe or estimate E_0, σ_E, D and T, and use (1) and (2) to solve for A_0 and σ_A

Valuing debt

- Default event is defined at *T* by
  - Bond holders get less than *D,* or
  - Equity holders get nothing, or
  - $A_T < D$
- The probability of default is

$$\mathcal{P}\{A_T < D\} = N(-d_2)$$

- The value of the debt comes from the parity relationship:

$$A_t = E_t + B_t$$

- To apply to arbitrary debt instruments, we can be a bit sloppy and price using the default probability and the Hull-White framework from before.

Issues with Merton

- Relationships between equity and credit markets come and go
- Measuring D is a matter of art.
- Assuming a single debt maturity is controversial.
- What about the possibility of defaults before T?
- How to estimate equity volatility?
- Is the model for asset values well specified?

Extensions

- Pricing equity options consistently in the model framework enables use of implied volatility
  - Geske (1977) – compound option framework
  - Hull et al (2004), Stamicar and Finger (2005) – calibrate to one or more option quotes
- Further extension enables the treatment of the default barrier (D) as endogenous (implied)
  - Leland and Toft (1996)
  - Hull et al (2004)
  - Stamicar and Finger (2005)
- Calibrate to empirical default distribution
  - Crosbie and Bohn (2003)
- Distinction of long- and short-term debt
  - Geske (1977)
- Default timing, first passage; random or non-constant barriers; random interest rates
  - See references in Fabozzi et al (2010)

Example -- MSCI

- Balance sheet as of 31 Aug 2010 (Yahoo Finance)
  - Total liabilities – $1914.26M
  - Total current liabilities – $409.32M
  - Shares outstanding – 118.56M
- Do current liabilities contribute to long-term leverage? Common practice is to count only 50%.
- So total effective liabilities comes to 0.5*$409.32+(1914.26-409.32) = $1709.60M
- In the model, it is convenient to express everything on a per share basis, so D = $14.42

Example -- MSCI

- Stock price (30 Nov 2010) = 34.78. Assume discount rate 2%
- Annualized volatility based on one year of daily stock returns: 32%
- Options (Yahoo Finance, 30 Nov 2010), expiry 19 Mar 2011
  - Put struck at 35: Price=2.6, BS Volatility=34.1%
  - Put struck at 30: Price=1.05, BS Volatility=40.8%
- Options (Yahoo Finance, 30 Nov 2010), expiry 17 Jun 2011
  - Put struck at 30: Price=1.45, BS Volatility=36.0%
  - Put struck at 25: Price=0.7, BS Volatility=43.6%
- Recall that σ_E in the model is *not* the Black-Scholes volatility. We are making an approximation in using these figures without adjustment.

Example -- MSCI

- Assume debt matures at five years.
- So start with

  E_0= 34.78, σ_E= 32%, T= 5, D= 14.42
- Initial guess: A_0= 40, σ_A= 25% (what are constraints on these?)
- Solving (1) and (2) iteratively gives

  A_0= 47.80, σ_A= 23.4%
- Implied probability of default is P=1.31%

Interpreting results

- To compare with Z-Metrics, annualize the default probability, that is, find *p* such that

$$1 - P = (1 - p)^T$$

- Annualized default probability is p= 0.26%. One-year PD from Z-Metrics is 0.10%.
- Asset value converted from per share is \$5667M. Balance sheet total assets is \$1200M.
- Traditional leverage ratio is A_0/E_0= 1.37

Volatility sensitivity

σ_E	A_0	σ_A	P	p	Leverage
32.0%	47.80	23.4%	1.31%	0.26%	1.374
34.1%	47.78	24.9%	2.03%	0.41%	1.374
36.0%	47.76	26.4%	2.86%	0.58%	1.373
40.8%	47.66	30.1%	5.64%	1.15%	1.370
43.6%	47.57	32.4%	7.71%	1.59%	1.368

- Across the range of plausible volatility values, leverage and asset value are stable, but default probability changes by a factor of six.

PORTFOLIO MODELS

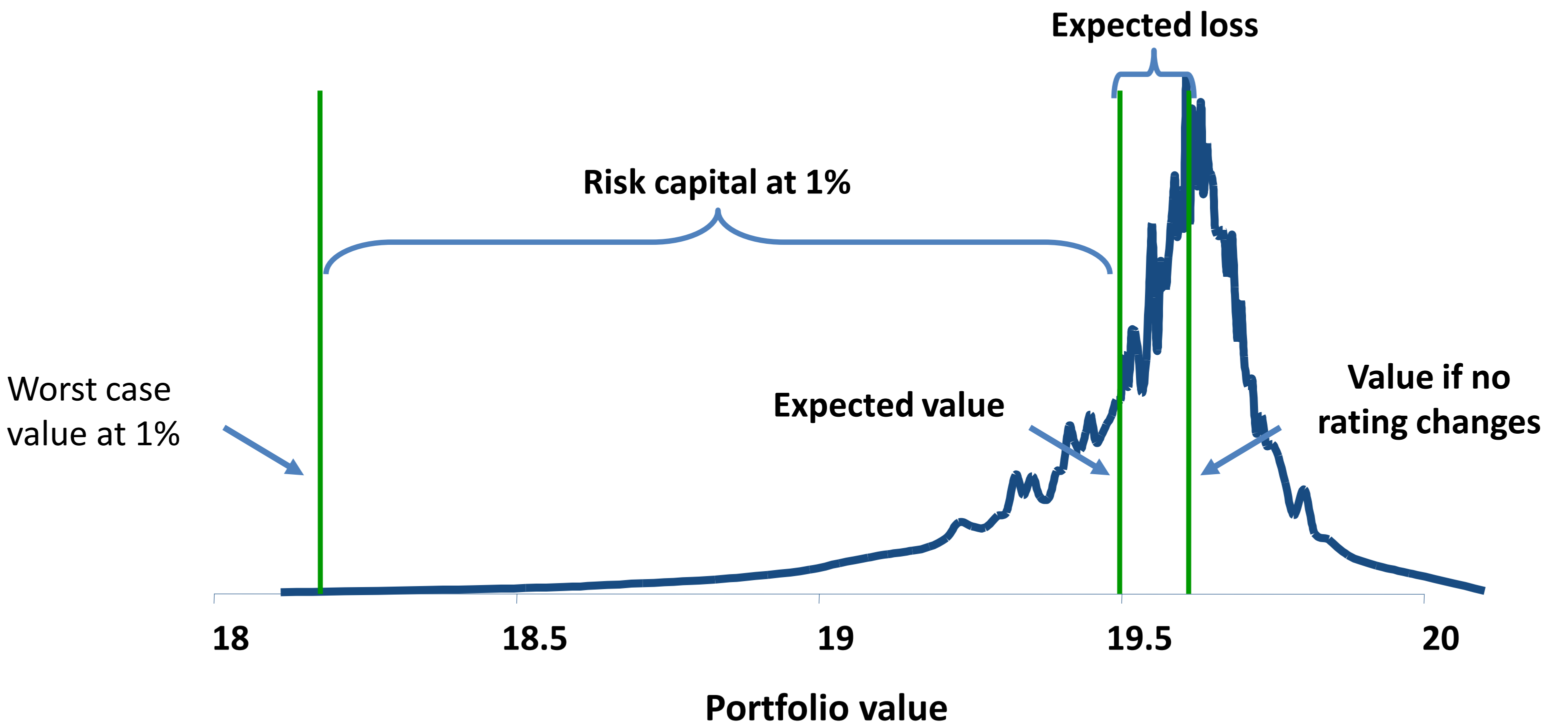
Agenda

- The CreditMetrics model – single horizon portfolio credit risk
- The Vasicek distribution
- Extending to continuous time (sort of) – the copula framework and CDO pricing
- Sources of correlation data

Two major applications

- Assess the economic capital required to support a portfolio of defaultable but possibly not tradeable positions over long time horizons, specifically one year.
- Price derivative and structured products (e.g CDOs) referencing a portfolio of defaultable securities.

What is economic capital?



- Expected loss is typically covered through pricing or reserves, and is additive.
- Capital is covered by the firm's equity (or other loss buffers), and is in general not simply additive.

How to build a portfolio model?

- We need correlations, but of what?
- Consider random variables X_1 and X_2 defined as default indicators:

$$X_i = \begin{cases} 1, \text{if obligor } i \text{ defaults}, \\ 0, \text{otherwise} \end{cases}$$

- Let p_i be the default probability for obligor *i,* and p_{12} be the joint default probability.
- The default correlation ρ_D is defined as the correlation coefficient between X_1 and X_2:

$$\rho_D = \frac{p_{12} - p_1 p_2}{\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}}$$

Are default correlations enough?

- For a general portfolio with *n* obligors, is the model fully specified with p_i for each obligor and ρ_D for all pairs?
- Consider *n*=3. There are eight distinct events (each obligor can default or not), plus one constraint (probabilities have to sum to one).
- But the p_i and ρ_D only give us six parameters.
- The model needs more structure.

A simple model

- Assume p_i are given for all obligors for a particular time horizon.
- Inspired by the Merton framework, assume each obligor's default is driven by its asset value Z_i.
- So obligor *i* defaults if $Z_i < \alpha_i$, where α_i is the default threshold.

- Now assume each Z_i follows a standard normal distribution. Then default thresholds are given by $\alpha_i = \Phi^{-1}(p_i)$.

- Assume the Z_i jointly follow a multivariate normal distribution. Then since the default distribution depends only on the Z_i, the model is fully specified.

- We refer to the correlations between the Z_i as *asset correlations*.
- Since the p_i are given exogeneously, the model is insensitive to the mean and variance of Z_i.

Is this model sufficient for capital?

- Future portfolio value is given by
$$V_P = \sum_i E_i X_i$$
where E_i is the exposure to obligor *i* and X_i is the default indicator.
- The model distinguishes between good and bad credits, and large and small positions.
- The model is sensitive to industry and sector concentrations through the correlation parameters.
- But the model is insensitive to maturity.

Capturing maturity risks

- We would like to capture the effect that a longer maturity instrument has more time to get into trouble.
- One approach is to let the p_i in the model refer to the default probability over the life of the bond or load in question, but …
  - The portfolio loss is an odd definition, with no specific timing, and
  - Correlation is odd, in that events that occur over different periods are still correlated.
- The alternate approach is to maintain a fixed horizon, but reflect credit quality changes other than default, namely ratings migrations.
- Migration probabilities typically derive from transition matrices, as discussed previously.

Capturing maturity risks

- Assume bonds and loans are valued according to spreads that depend on their rating and (possibly) maturity.
- A rating change will trigger a revaluation according to a different spread, for instance using the OAS formula from earlier.
- Longer maturity instruments will be more sensitive to these changes in spread.

	Maturity				
	2	3	5	7	10
AAA	15	20	25	27	30
AA	20	25	30	32	37
A	30	40	45	50	58
BBB	62	68	75	80	85
BB	200	225	250	300	300
B	300	350	375	400	400
CCC	1100	950	775	725	700

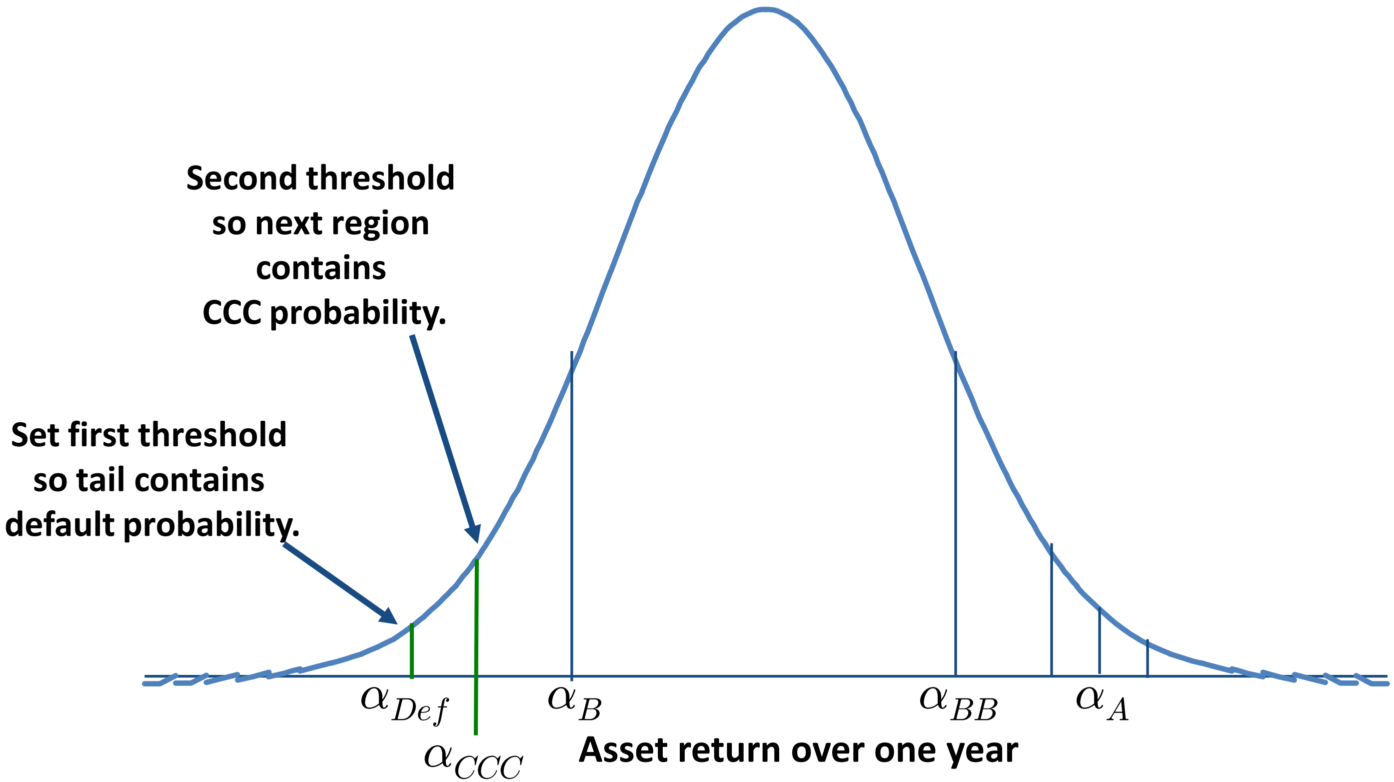
The new univariate distribution

Current state				Baa				
Possible states at horizon	Aaa	Aa	A	Baa	Ba	B	Caa	Default
Probabilities	0.00%	0.11%	5.28%	86.71%	6.12%	1.27%	0.23%	0.28%

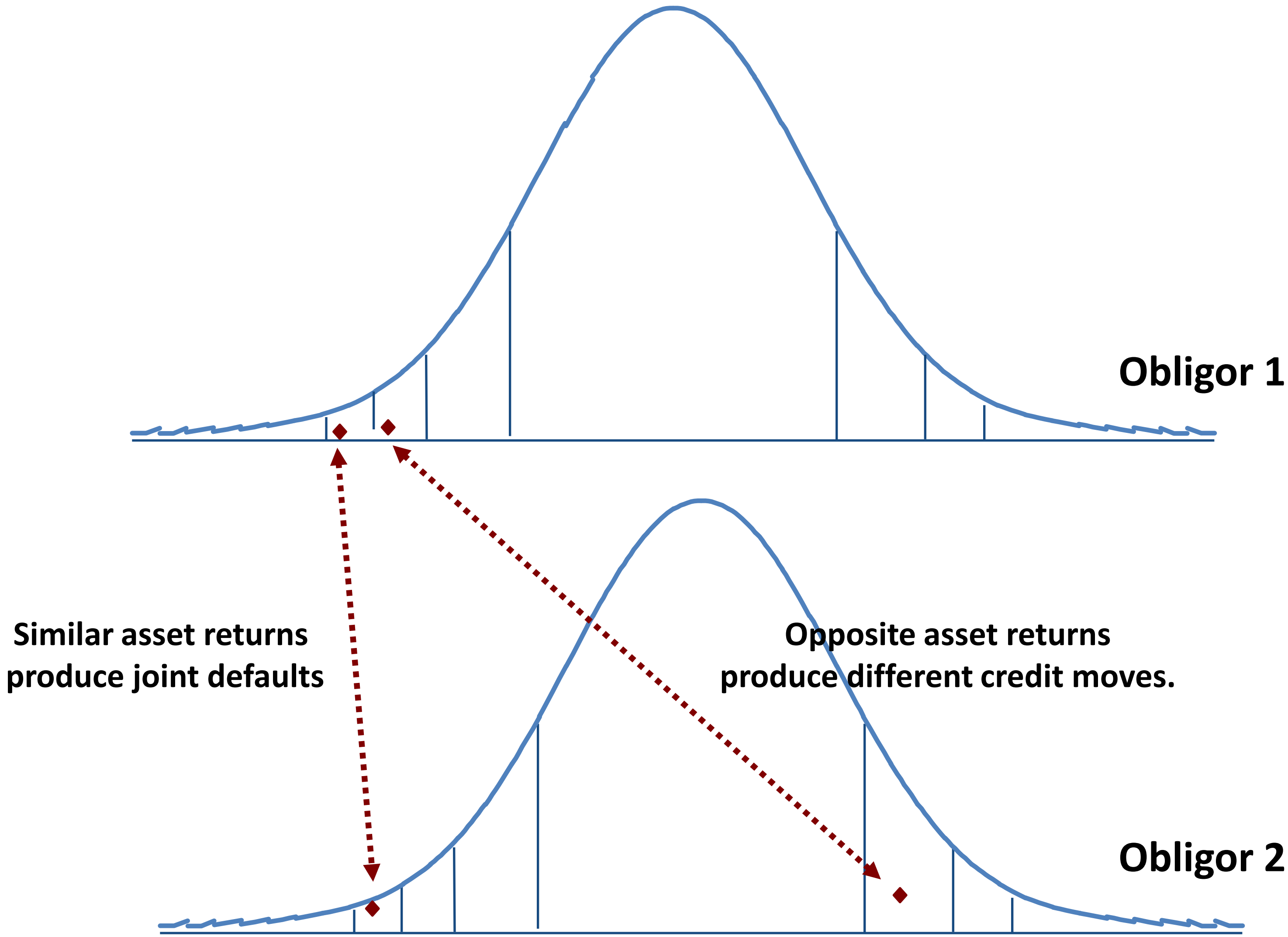
(determined exogenous to model)

Instrument value	100.9%	100.8%	100.7%	Par	97.5%	95.8%	83.2%	Rec.

Generalize the two-state model



One correlation gives all joint probabilities



Most typically, the model is implemented through Monte Carlo

1. Compute default and transition thresholds – α_{Def}, α_{CCC}, etc. – for each obligor, according to each obligor's rating.
2. Generate random variables Z_i corresponding to each obligor, according to the asset correlation matrix.
3. Compare each Z_i to the obligor thresholds, and assign a rating.
4. Compute the value of the positions for each obligor according to the new rating.
5. Aggregate to get a portfolio value.
6. Repeat *N* times, and compute risk statistics.

A special case

- Return to the two-state model.
- Assume that the correlation structure is generated by a *single-factor model*:

$$Z_i = \sqrt{\rho} Z + \sqrt{1-\rho}\varepsilon_i$$

where

- ρ is a single pairwise correlation parameter,
- *Z* is a common market factor, distributed N(0,1), and
- ε_i are idiosyncratic to the individual obligors, distributed N(0,1), mutually independent and independent of *Z*

Conditional default probability

- Recall the default threshold for each obligor is
$$\alpha_i = \Phi^{-1}(p_i)$$
where p_i is the unconditional default probability.
- Define the conditional default probability
$$p_i(z) \equiv \mathcal{P}\{\text{Obligor } i \text{ defaults} | Z = z\}$$
- The default condition is
$$Z_i < \alpha_i \iff \varepsilon_i < \frac{\alpha_i - \sqrt{\rho} Z}{\sqrt{1-\rho}}$$

Two important observations

- The conditional default probability is

$$\begin{aligned} p_i(z) &= \mathcal{P}\left\{Z_i < \alpha_i | Z = z\right\} \\ &= \mathcal{P}\left\{\varepsilon_i < \frac{\alpha_i - \sqrt{\rho} z}{\sqrt{1-\rho}}\right\} \\ &= \Phi\left(\frac{\alpha_i - \sqrt{\rho} z}{\sqrt{1-\rho}}\right) \end{aligned}$$

- Conditional on *Z*, all obligor defaults are independent.

The fine-grained limit

- Assume
  - All default probalities are equal … $p_i \equiv p$
  - There are *N* obligors in the portfolio.
  - The exposure to each obligor is *1/N*.
  - Recovery is zero for each obligor.
  - The correlation structure is as defined previously.
- Portfolio loss is $$L = \frac{1}{N}\sum_{i=1}^{N} X_i$$
- We are interested in the portfolio distribution in the limit as $N \rightarrow \infty$.

Condition on the market factor … *Z=z*

- Each obligor has conditional default probability p(z).
- Obligor defaults are conditionally independent.
- The number of obligor defaults conditionally follows a binomial distribution with parameters p(z) and N.

Condition on the market factor … *Z=z*

- Conditional mean of portfolio loss

$$\mathbf{E}[L|Z=z] = \frac{1}{N}\sum_{i=1}^{N}\mathbf{E}[X_i|Z=z] = p(z)$$

- Conditional variance of portfolio loss

$$\begin{aligned}
\mathbf{Var}[L|Z=z] &= \frac{1}{N^2}\mathbf{Var}\left[\sum_{i=1}^{N} X_i|Z=z\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\mathbf{Var}\left[X_i|Z=z\right] \text{ (using conditional independence)} \\
&= \frac{1}{N^2}\sum_{i=1}^{N} p(z)(1-p(z)) \\
&= \frac{p(z)(1-p(z))}{N}
\end{aligned}$$

In the limit N$\to\infty$

- Conditional on *Z=z*
  - Portfolio mean approaches *p(z)*
  - Portfolio variance approaches zero
- Moreover, $\lim_{z\to\infty} p(z) = 1, \ \lim_{z\to-\infty} p(z) = 0$
- So $\lim_{z\to\pm\infty} \mathbf{Var}[X_i | Z = z] = \lim_{z\to\pm\infty} p(z)(1 - p(z)) = 0$
- Portfolio loss converges to

$$L \to p(Z) = \Phi\left(\frac{\alpha - \sqrt{\rho} Z}{\sqrt{1-\rho}}\right)$$

The limiting distribution

- In the limit, the portfolio loss is distributed as p(Z), where Z is distributed N(0,1).
- This distribution is referred to as the *fine-grained limit* or the *large pool model* or the *Vasicek distribution*.
- The portfolio CDF is easy to compute:

$$
\begin{aligned}
\mathcal{P}\{L < l\} &= \mathcal{P}\left\{\Phi\left(\frac{\alpha - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) < l\right\} \\
&= \mathcal{P}\left\{\frac{\alpha - \sqrt{\rho}Z}{\sqrt{1-\rho}} < \Phi^{-1}(l)\right\} \\
&= \mathcal{P}\left\{Z > \frac{\alpha - \sqrt{1-\rho}\Phi^{-1}(l)}{\sqrt{\rho}}\right\} \\
&= \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(l) - \alpha}{\sqrt{\rho}}\right)
\end{aligned}
$$

PRICING STRUCTURED CREDIT

Expanding to multiple horizons

- The applications discussed so far concern the portfolio distribution at a single horizon.
- Other applications need to address multiple horizons, and to deal with default timing explicitly.
- One important example is the pricing of *collateralized debt obligations* (CDOs).

Example: synthetic CDO in swap form

- Define a reference portfolio containing, for example, unit positions on 100 obligors.
- Protection seller compensates for losses on the index in excess of one level (the attachment point) and up to a second level (the detachment point).
- For example, on the 3-7% tranche of the CDX, protection seller pays losses over 3% (attachment) and up to 7% (detachment).
- Protection buyer pays an upfront amount (for most junior tranches) plus a periodic premium (for example, 100bp) on the remaining amount being protected.
- Pricing depends on the distribution of losses on the index, not just the expectation.

Example: synthetic CDO in swap form

- Assume the swap is for five years, with annual payments.
- Assume total portfolio notional is 2500. Notional amount protected is (7%-3%)*2500= 100.
- First year
  - Two defaults with recovery 40% each … loss of 1.2% of original portfolio
  - Protection seller pays nothing (losses have not reached attachment point)
  - Protection buyer pays 100bp*100=1 (payment based on amount protected for the period)

Example: synthetic CDO in swap form

- Second year
  - Five defaults with recovery 40% each … loss of 3% of original portfolio … cumulative losses now 4.2%.
  - Protection seller pays (4.2%-3%)*2500=30.
  - Protection buyer pays 100bp*100=1.
  - Notional protected for next period is (7%-4.2%)*2500=70.

Example: synthetic CDO in swap form

- Third year
  - Zero defaults … cumulative losses still 4.2%.
  - Protection seller pays zero.
  - Protection buyer pays 100bp*70=0.7.
  - Notional protected for next period is still 70.
- Fourth year
  - One default … cumulative losses now 4.8%.
  - Protection seller pays 0.6%*2500=15.
  - Protection buyer pays 100bp*70=0.7.
  - Notional protected for next period is 55.

Example: synthetic CDO in swap form

- Fifth year
  - Ten defaults … cumulative losses now 10.8%
  - Protection seller pays (7%-4.8%)*2500=55.
  - Protection buyer pays 100bp*55=0.55.
  - Notional protected for next period would be zero.

Pricing – general framework

- Risk-free discounting from t – D(t)
- Spread rate – s
- Attachment (a) and detachment (d) points
- Assume original total portfolio is 1.
- Cumulative portfolio loss to time t – L_t
- Cumulative tranche loss to t –

$$TL_t = \min\{d - a, \max\{L_t - a, 0\}\}$$

- Remaining notional protected –

$$N_t = (d - a) - TL_t$$

Pricing – general framework

- Expectation of discounted protection buyer payments

$$s \sum_{t=1}^{T} D(t) \mathbf{E} N_{t-1}$$

- Expectation of discounted protection seller payments

$$\sum_{t=1}^{T} D(t) \mathbf{E} [TL_t - TL_{t-1}]$$

- Mark-to-market of a specific tranche is the difference of these.
- Fair spread is the value of s that makes the Mark-to-Market equal zero.
- See Mina and Stern (2003).

Pricing – model needs

- The expectations inside the pricing boil down to terms that look like

$$\mathbf{E}\min\{d-a,\max\{L_t-a,0\}\}$$

- This depends on the distribution of L_t at each t, but not on the joint distribution of L_t and L_s.
- It suffices to specify the joint distribution of the obligor default times T_i.
- Other models exist, but we will focus on the classic one.

Gaussian copula model (Li, 2001)

- We know the univariate distributions for the T_i from our CDS pricing. Let $P_i(t) = \mathcal{P}\{T_i < t\}$
- Assume the T_i are driven by standard normal random variables Z_i, such that

$$\begin{aligned} U_i &= \Phi(Z_i) \\ T_i &= P_i^{-1}(U_i) \end{aligned}$$

- Correlations among Z_i induce dependence across T_i.
- Li (2001) shows that restricted to a single period, this is identical to the CreditMetrics model, meaning that we may use similar sources of correlation information.

Applications

- We may use the model to simulate T_i under arbitrary correlation structures, in order to evaluate any product depending on portfolio default losses.
- Under the one-factor correlation structure, we may use the conditioning argument to reduce many problems to a one-dimensional numerical integral.
- For large, homogeneous portfolios, we may use the large pool model as our distribution for L_t.

General framework for the one-factor case

- We wish to compute something like

$$\mathbf{E}\min\{d-a, \max\{L_t - a, 0\}\}$$

where

$$L_t = \sum_i I_{T_i<t} \qquad \text{(portfolio loss is the sum of default indicators)}$$

$$T_i = P_i^{-1}(\Phi(Z_i)) \qquad \text{(default driven by normal random variables)}$$

$$T_i < t \iff Z_i < \Phi^{-1}(P_i(t)) \qquad \text{(reduction to single period case)}$$

$$Z_i = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon_i \qquad \text{(one-factor correlation structure)}$$

General framework for the one-factor case

1. **Conditioning step**. Condition on the market factor Z. Compute conditional default probabilities.
2. **Convolution step**. Use conditional independence to compute (or approximate) conditional distribution of L_t. This could mean using the binomial distribution explicitly (for small portfolios) or applying Fourier transforms or approximating with the large pool model. Calculate the conditional expectation of the min-max form desired.
3. **Integration step**. Numerically integrate over the different values for Z, using the Gaussian density for Z.

CORRELATION DATA

Asset and equity correlation

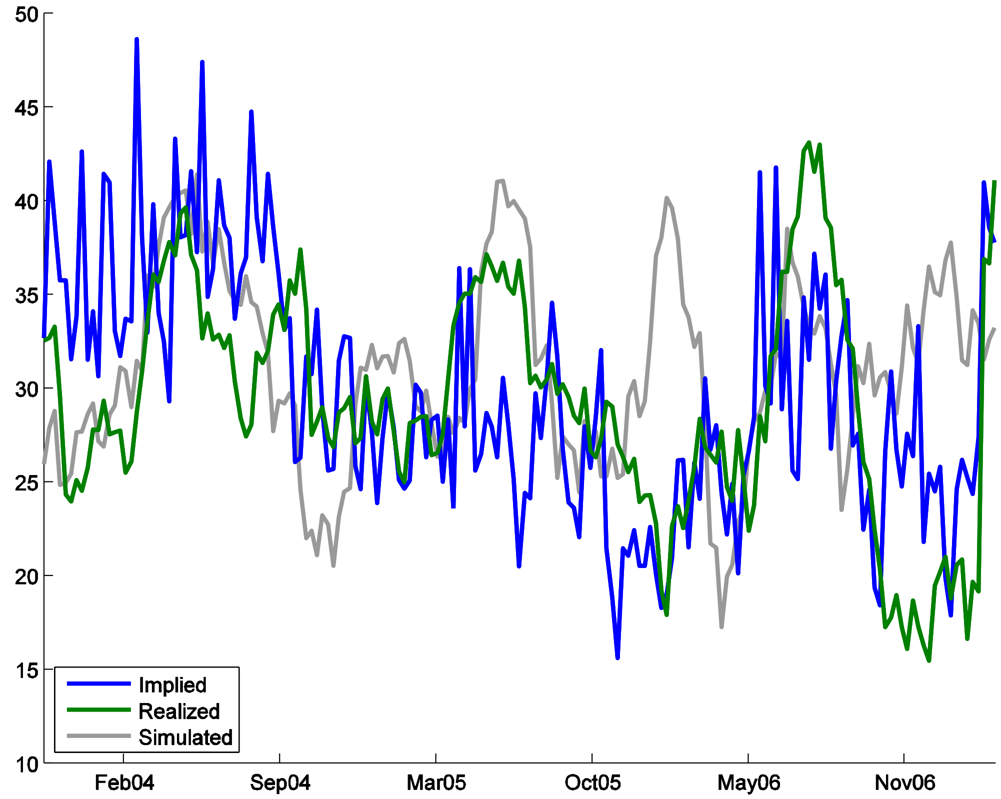
- Under the Merton model, equity is a function of the asset value. We used Ito's lemma before to derive the adjustment in instantaneous volatility.
- The same argument implies that for two firms, the instantaneous asset *correlation* is the same as the instantaneous equity correlation.
- This means we can use observable equity correlations to parameterize the portfolio model.

Empirical equity correlations (700 firms)

- Five years of weekly data:
  - 25% within industries
  - 16% across
- One year of daily data:
  - 31% within industries
  - 22% across

Correlations across the Dow 30

- Realized correlation over rolling 50-day periods, averaged across all pairs
- Implied correlation from options on index and options on individual stocks



Empirical default data

- Characterize correlation through default rate volatility, using the mapping defined earlier.
- De Servigny and Renault (2003)
  - Default correlation: 5.3% within industry, 1.3% across
  - Asset correlation: 18% within industry, 6% across
  - *But* sampling error could be as high as 50%
- Demey et al (2004)
  - Maximum likelihood, estimating correlations directly but on empirical default data
  - Correlations within industry 10-15% at low end (consumer, auto, tech), 25-40% at high end (insurance, real estate)
  - Correlations across industry 7-10%
- This approach is also useful for retail and SME portfolios, where empirical default data is more likely available than equity data.

Basel II

- Correlations for corporates depend on the PD, set by policy

$$\rho = 0.12 \frac{1 - \exp(-50p)}{1 - \exp(-50)} + 0.24 \frac{\exp(-50p)}{1 - \exp(-50)}$$

- Maturity adjustment calibrated to empirical examples using credit migrations

$$b = (0.11852 - .05478 \cdot \log(p))^2$$

- IRB formula uses the Vasicek distribution.

$$(1 - R) \cdot \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(p) + \sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(0.999)\right) \cdot \frac{1 - 1.5b}{1 + (M - 2.5)b}$$

Rating agencies – Standard & Poor’s

- CDO Evaluator used to establish critical default level for desired rating.
- Monte Carlo is performed according to the asset value model.
- Correlation assumptions (corporates):
  - 30% within industry
  - 0% across industry

Rating agencies – Moody's

- Old Diversity Score model -- characterize a correlated portfolio by a smaller independent portfolio.
- Variance of *n* correlated assets:

$$\frac{p(1-p)}{n}(1+(n-1)\rho_D)$$

- Variance of *N* independent assets:

$$\frac{p(1-p)}{N}$$

- Moody's tables are consistent with a default correlation of 16% within industry, 0% across.

Rating agencies – Moody's

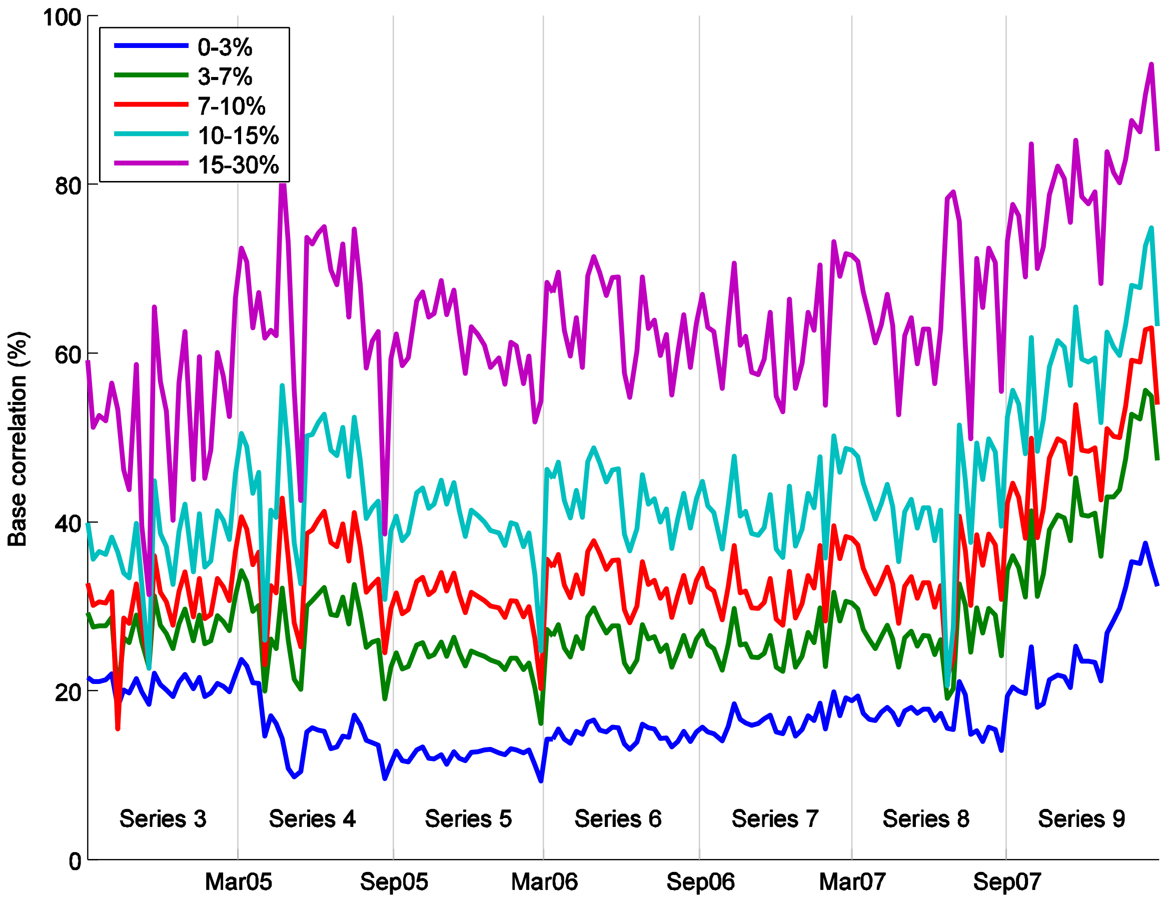
- Mapping from default to asset correlation depends on default rate.

	1	2	3	4	5	6	7	8	9	10
A1	95	79	72	70	68	66	65	64	63	61
A2	88	75	69	66	64	63	62	60	59	57
A3	78	71	66	63	61	60	58	57	56	54
Baa1	74	68	63	60	58	56	55	53	52	51
Baa2	70	64	60	57	55	53	51	50	49	47
Baa3	65	58	54	52	49	47	46	44	44	42
Ba1	60	53	49	46	44	42	41	40	39	38
Ba2	55	48	44	42	39	38	37	36	35	35
Ba3	50	43	40	38	36	35	34	33	33	33
B1	45	39	36	34	33	32	32	31	30	30
B2	41	36	33	32	31	30	29	29	29	29
B3	36	32	31	30	29	28	28	28	27	27
Caa1	32	30	29	28	28	27	26	26	26	26

- Broadly, an average asset correlation of 30-50% is used.

SCDO implied correlation

- Tranches on CDX North America



Comparison of Correlation Data Sources

- Equity returns, US firms, 1998-2002
  - Average intra-industry correlation = 25%
  - Overall average correlation = 17%
- Implied equity correlations
  - Across DJX constituents … 20-40%
- Default history
  - S&P (1981-2001) … Intra-industry 18%, overall 7%
  - S&P (1981-2002) … MLE approach … Intra-industry 10-40%, overall 8-10%
- BIS IRB formula
  - Overall 10-20%
- CDO rating models
  - Intra-industry 30-40%, independent across
- Synthetic CDO pricing
  - CDX NA equity correlation stable around 20% until 2008, now about 30%
  - Higher values for more senior tranches

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