Fixed Income and Credit Risk : solutions for exercise sheet n° 06

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Professor	Assistant	Program
Fulvio Pegoraro	Roberto Marfè	MSc. Finance

Exercise N° 01 [Floating Rate Bond (from Veronesi (2010))].

Given the relation between the coupon bond and zero-coupon bond prices, we can write:

$$CB(0.25, 0.5) = B(0.25, 0.5)\, 100 \left(1 + \frac{0.03}{4}\right) = 100.0448\,,$$

and, therefore, the proper discount factor is $B(0.25, 0.5) = 0.9930$. This means that, the price of the floating rate bond is:

$$CB_{FR}(0.25, 6) = B(0.25, 0.5)\, 100 \left(1 + \frac{0.03}{2}\right) = 100.7895\,.$$

Exercise N° 02 [Floating Rate Bond (from Veronesi (2010))].

Give the information we obtain from the market, we have that the one-quarter discount factor (i.e., the zero-coupon bond price for a unitary face value) is $B(0, 0.25) = 0.9980$. In addition, given the representation of the coupon bond price as a portfolio of zero-coupon bonds we can write:

$$100.3960 = 0.9980 \times \frac{0.02}{4} \times 100 + B(0, 0.5) \times 100 \times (1 + 0.02/4)\,,$$

we easily obtain $B(0, 0.5) = 0.994$. With these discount factors in hand, we can calculate the price of the floating rate bond thanks to the following formula:

$$\begin{aligned} CB_{FR}(0, 0.5) &= 100 + s \times 100 \times [B(0, 0.25) + B(0, 0.5)] \\ &= 100 + \frac{0.01}{4} \times 100 \times (0.9980 + 0.9940] = 100.498\,. \end{aligned}$$

Exercise N° 03 [Forward Rate Agreements with opposite payoffs (from Veronesi (2010), chapter 5)].

We have that the payoff of reverse FRA at τ is $\frac{N}{2} \times [Y^{(2)}(\tau, T) - Y^{(2)}(s, \tau, T)]$. Clearly, the payoff of the FRA at time s depends on the current forward rate $Y^{(2)}(s, \tau, T)$ instead of the old one

$Y^{(2)}(t,\tau,T)$. The total payoff for the firm at T is then:

$$\begin{aligned}\text{Total payoff at } T \text{ of old and new } FRAs &= \left\{\frac{N}{2} \times \left(Y^{(2)}(t,\tau,T) - Y^{(2)}(\tau,T)\right)\right\} \\ &\quad + \left\{\frac{N}{2} \times \left(Y^{(2)}(\tau,T) - Y^{(2)}(s,\tau,T)\right)\right\} \\ &= \frac{N}{2} \times [Y^{(2)}(t,\tau,T) - Y^{(2)}(s,\tau,T)]\,,\end{aligned}$$

that is, at the time of the decision to close the original FRA contract, the firm will end up with a positive payoff if the current forward rate declined since the inception, and vice versa. Since $B(s,\tau) = \$99.10$ and $B(s,T) = \$97.37$, the current forward rate is $Y^{(2)}(s,\tau,T) = 2 \times (0.991/0.9737 - 1) = 3.55\% < Y^{(2)}(t,\tau,T) = 4.21\%$. Assuming $N = \$100\text{m}$, the time T payoff is then $\frac{N}{2} \times [Y^{(2)}(t,\tau,T) - Y^{(2)}(s,\tau,T)] = \$328,272$ which is known at the earlier time s. Therefore, the present value is:

$$V^{fixed}(s) = B(s,T) \times \frac{N}{2} \times [Y^{(2)}(t,\tau,T) - Y^{(2)}(s,\tau,T)] = 0.9737 \times \$328,272 = \$319,638\,,$$

as we obtained at the end of Example 3.

Exercise N° 04 [Equivalence between Forward contract return and forward rate (from Veronesi (2010), chapter 5)].

i) From Lecture 6 (slide 55) we can write:

$$\Phi(t,\tau,T) = \Phi(t, t+0.5\,\text{years}, t+1\,\text{year}) = \frac{B(t, t+1\,\text{year})}{B(t, t+0.5\,\text{years})} = 100 \times \frac{0.95713}{0.97728} = \$97.938\,.$$

So, the forward price at date t for the investment, between $t+0.5$ years and $t+1$ year, on the 6-months T-bill with face value \$100 is $\Phi(t, t+0.5\,\text{years}, t+1\,\text{year}) = \$97.938$ (equal to the forward discount factor $F(t, t+0.5\,\text{years}, t+1\,\text{year})$).

ii) We know that the the firm enter into a forward contract to purchase $M = 1.02105$ million of 6-month T-bill on τ (with \$100 of par value). The payoff from the forward contract at time τ is then given by:

$$\text{Payoff forward contract at } \tau \;=\; M \times [B(\tau,T) - \$97.938]\,.$$

Ex post, the fear of the firm that the interest rate would decline is in fact realized, and the price of the 6-month T-bill at $\tau = t+0.5$ years turns out to be $B(t+0.5\,\text{years}, t+1\,\text{year}) = \$98.89 > \$97.938$. Thus, the payoff from the forward contract is:

$$\text{Payoff forward contract at } \tau \;=\; 1.02105\text{m} \times [\$98.89 - \$97.938] = \$972,043.54\,.$$

The firm at $\tau = t + 0.5$ years can then invest this additional amount, \$972, 043.54, together with the receivable \$100, 000, 000 into the new T-bill. In particular, it will be purchasing an amount of T-bill equals to:

$$\text{Investment in T-bills at } \tau \;=\; \frac{(\$100,000,000 + \$972,043.54)}{\$98.89} = 1,021,054.136\,,$$

where the T-bills have \$100 principal. At maturity $T = t + 1$ year, the total amount realized is then \$102,105,413.6, which implies a realized annualized rate of return equal to

$$\begin{aligned}\text{Annualized rate of return} &= \frac{1}{T-\tau} \times \left(\frac{\text{Payoff at } T}{\text{Investment at } \tau} - 1\right) \\ &= 2 \times \left(\frac{102,105,413.6}{100,000,000} - 1\right) = 0.0421\,.\end{aligned}$$

This is exactly equal to the forward rate determined in the Example 1 of Lecture 6, and is in fact independent of the realized T-bill price at τ, i.e. $B(t + 0.5\,\text{years}, t + 1\,\text{year}) = \$98.89$. The higher the price of the T-bill, the higher the payoff from the forward contract. But more it becomes more expansive to purchase T-bills for an investment between τ and T. These two effects exactly cancel each other out, and the firm is guaranteed the forward rate.

Exercise N° 05 [Forward on a Coupon Bond].

From Lecture 6, we know that:

$$\begin{aligned}\Phi^{CB}(t,T) &= \frac{E_t\left[M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}\; CB^*(T,\widetilde{T})\right]}{B(t,T)} \\ &= \frac{\sum_{i=1}^{n} C_i\; E_t\left[M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}\; B(T,T_i)\right]}{B(t,T)} \\ &= \frac{\sum_{i=1}^{n} C_i\; B(t,T_i)}{B(t,T)}\,.\end{aligned}$$

This means that the forward price is $\Phi^{CB}(t,T) = CB(t,\widetilde{T})/B(t,T) = 115/97 = 1.1856$.

Exercise N° 06 [Value of a Swap].

The value of the swap is given by:

$$\begin{aligned}& V_{swap}(0, 1.5; 5.52\%) \\ =\ & 100 - \left[\sum_{j=0.5}^{1.5} \frac{0.0552}{2} \times 100 \times B(0,j) + 100 \times B(0,1.5)\right] \\ =\ & 100 - \left[\frac{0.0552}{2} \times 100 \times 0.9745 + \frac{0.0552}{2} \times 100 \times 0.9490 + \left(\frac{0.0552}{2} + 1\right) \times 100 \times 0.9215\right] \\ =\ & 100 - 2.6889 - 2.6185 - 94.6926 = 0\,,\end{aligned}$$

and therefore the value of the swap is correctly equal to zero given that we are at the inception of the contract.

Exercise N° 07 [Exercise N° 06, continued].

The value of the swap is given by:

$$
\begin{aligned}
& V_{swap}(0.25, 1.5; 5.52\%) \\
= \; & B(0.25, 0.5) \times 100 \times \left(1 + \frac{r_2(0)}{2}\right) - \frac{0.0552}{2} \times 100 \times (B(0.25, 0.5) + B(0.25, 1) + B(0.25, 1.5)) \\
& \qquad -100 \times B(0.25, 1.5) \\
= \; & 98.40 \times \left(1 + \frac{0.0517}{2}\right) - \frac{0.0552}{2} \times 100 \times (0.9840 + 0.9520 + 0.9190) - 100 \times 0.9190 \\
= \; & 100.94 - 2.7151 - 2.6268 - 94.4357 = 1.16604\,.
\end{aligned}
$$

Exercise N° 08 [Defaultable ZCB price in a simple setting with general default date distribution].

a) The price of the defaultable ZCB at the beginning of the period is:

$$
\begin{aligned}
DB(0,T) &= E_0^{\mathbb{Q}}\left[\exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right) \left(\mathbb{I}_{\{\tau > T\}} + RR \times \mathbb{I}_{\{\tau \leq T\}}\right)\right] \\
&= E_0^{\mathbb{Q}}\left[\exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right) \left(1 - (1 - RR) \times \mathbb{I}_{\{\tau \leq T\}}\right)\right] \\
&= E_0^{\mathbb{Q}}\left[\exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right)\right] - (1 - RR)\, E_0^{\mathbb{Q}}\left[\exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right) \times \mathbb{I}_{\{\tau \leq T\}}\right] \\
&= B(0,T) - (1 - RR) \exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right) E_0^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau \leq T\}}\right] \\
&= B(0,T) - (1 - RR) \exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right) \mathbb{Q}(\tau \leq T) \\
&= B(0,T) - (1 - RR) \exp\left(-\textstyle\sum_{i=0}^{T-1} r_i\right) F^*(T)
\end{aligned}
$$

b) The price of the defaultable ZCB at $t \in (0,T)$, conditionally to $\tau > t$, is:

$$\begin{aligned}
DB(t,T) &= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{T-1} r_i\right)\left(\mathbb{I}_{\{\tau>T\}} + RR \times \mathbb{I}_{\{\tau\le T\}}\right) \mid \tau > t\right] \\
&= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{T-1} r_i\right)\left(1 - (1-RR) \times \mathbb{I}_{\{\tau\le T\}}\right) \mid \tau > t\right] \\
&= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{T-1} r_i\right) \mid \tau > t\right] - (1-RR)\, E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{T-1} r_i\right)\mathbb{I}_{\{\tau\le T\}} \mid \tau > t\right] \\
&= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{T-1} r_i\right)\right] - (1-RR)\exp\left(-\sum_{i=t}^{T-1} r_i\right)\mathbb{Q}(\tau \le T \mid \tau > t) \\
&= B(t,T) - (1-RR)\exp\left(-\sum_{i=t}^{T-1} r_i\right)\frac{\mathbb{Q}(t<\tau\le T)}{\mathbb{Q}(\tau>t)} \\
&= B(t,T) - (1-RR)\exp\left(-\sum_{i=t}^{T-1} r_i\right)\frac{F^*(T)-F^*(t)}{1-F^*(t)}\,.
\end{aligned}$$

Observe that the value of the defaultable zero-coupon $DB(t,T)$ is discontinuous at time $t = \tau$, except if $F(T) = 1$. In this case, the default appears with probability one before maturity.

c) Given that $\dfrac{F^*(T)-F^*(t)}{1-F^*(t)} = \mathbb{Q}(\tau \le T \mid \tau > t)$, this expression is the risk-neutral default probability conditionally to not being in default at t (i.e., $\tau > t$).

d) The general date t price $\widetilde{DB}(t,T)$ (say) of a defaultable ZCB can be written in the following way:

$$\widetilde{DB}(t,T) = \mathbb{I}_{\{\tau\le t\}}\, RR\, \exp\left(-\sum_{i=t}^{T-1} r_i\right) + \mathbb{I}_{\{\tau>t\}}\, DB(t,T)\,.$$

Observe that, when $t = 0$, we have $\widetilde{DB}(0,T) = DB(0,T)$.

Exercise N° 09 [Exercise N° 08, continued].

i) The price of the defaultable ZCB at the beginning of the period is:

$$\begin{aligned}
DB(0,T) &= E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(s)\,ds\right)\left(\mathbb{I}_{\{\tau>T\}} + RR(\tau) \times \mathbb{I}_{\{\tau\le T\}}\right)\right] \\
&= \exp\left(-\int_0^T r(s)\,ds\right) E_0^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau>T\}}\right] + \exp\left(-\int_0^T r(s)\,ds\right) E_0^{\mathbb{Q}}\left[RR(\tau) \times \mathbb{I}_{\{\tau\le T\}}\right] \\
&= \exp\left(-\int_0^T r(s)\,ds\right)\mathbb{Q}(\tau > T) + \exp\left(-\int_0^T r(s)\,ds\right)\int_0^T RR(s)\, f^*(s)\, d\,s\,.
\end{aligned}$$

ii) The price of the defaultable ZCB at $t \in (0,T)$, conditionally to $\tau > t$, is:

$$
\begin{aligned}
DB(t,T) &= E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)\left(\mathbb{I}_{\{\tau>T\}} + RR(\tau)\,\times\,\mathbb{I}_{\{\tau\le T\}}\right)\,|\,\tau>t\right] \\
&= \exp\left(-\int_t^T r(s)\,ds\right)\mathbb{Q}(\tau>T\,|\,\tau>t) + \exp\left(-\int_t^T r(s)\,ds\right)E_t^{\mathbb{Q}}\left[RR(\tau)\,\times\,\mathbb{I}_{\{\tau\le T\}}\,|\,\tau>t\right] \\
&= \exp\left(-\int_t^T r(s)\,ds\right)\frac{\mathbb{Q}(\tau>T)}{\mathbb{Q}(\tau>t)} + \exp\left(-\int_t^T r(s)\,ds\right)\frac{1}{\mathbb{Q}(\tau>t)}\int_t^T RR(s)\,f^*(s)\,d\,s\,.
\end{aligned}
$$

iii) To summarize:

$$
\widetilde{DB}(t,T) = \mathbb{I}_{\{\tau\le t\}}\,RR(\tau)\,\exp\left(-\int_t^T r(s)\,ds\right) + \mathbb{I}_{\{\tau>t\}}\,DB(t,T)\,.
$$

Exercise N° 10 [Defaultable ZCB price in a simple setting, intensity function and credit spread].

The default probability $\mathbb{P}(\tau > t)$ can be written as:

$$
\mathbb{P}(\tau>t) = 1 - F(t) = \exp(-\Lambda(t)) = \exp\left(-\int_0^t \lambda(s)\,d\,s\right),
$$

and the hazard (intensity) rate can be interpreted as the probability that default occurs in the infinitely small time interval $(t, t+dt]$, given that the default did not occur before time t. Indeed, let us first consider the time interval $(t, t+h]$ and let assume $\tau > t$. The probability that the default occurs in $(t, t+h]$, conditionally to $\tau > t$ is:

$$
\begin{aligned}
\mathbb{P}[\tau \le t+h\,|\,\tau>t] &= \frac{\mathbb{P}[t<\tau\le t+h]}{\mathbb{P}[\tau>t]} \\
&= \frac{F(t+h)-F(t)}{1-F(t)}\,.
\end{aligned}
$$

Now, if we compute $\lim_{h\to 0}\mathbb{P}[\tau \le t+h\,|\,\tau>t]/h$ we immediately obtain:

$$
\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\,\mathbb{P}[\tau\le t+h\,|\,\tau>t] &= \lim_{h\to 0}\frac{1}{1-F(t)}\,\frac{F(t+h)-F(t)}{h} \\
&= \frac{1}{1-F(t)}f(t) = \lambda(t)\,.
\end{aligned}
$$

The price of the defaultable ZCB at $t \in (0,T)$, conditionally to $\tau > t$, can be written as:

$$\begin{aligned}
DB(t,T) &= E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)\left(\mathbb{I}_{\{\tau>T\}} + RR \times \mathbb{I}_{\{\tau\leq T\}}\right) \mid \tau > t\right] \\
&= \exp\left(-\int_t^T r(s)\,ds\right) E_t^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau>T\}} \mid \tau > t\right] + RR\,\exp\left(-\int_t^T r(s)\,ds\right) E_t^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau\leq T\}} \mid \tau > t\right] \\
&= \exp\left(-\int_t^T r(s)\,ds\right) \frac{1-F(T)}{1-F(t)} + RR\,\exp\left(-\int_t^T r(s)\,ds\right) \frac{F(T)-F(t)}{1-F(t)}\,.
\end{aligned}$$

Now, given that $\dfrac{1-F(T)}{1-F(t)} = \exp\left(-\int_t^T \lambda(s)\,ds\right)$ and $\dfrac{F(T)-F(t)}{1-F(t)} = 1 - \exp\left(-\int_t^T \lambda(s)\,ds\right)$, we can write:

$$\begin{aligned}
DB(t,T) &= \exp\left[-\int_t^T (r(s)+\lambda(s))\,ds\right] + RR\,\exp\left(-\int_t^T r(s)\,ds\right)\left[1 - \exp\left(-\int_t^T \lambda(s)\,ds\right)\right] \\
&= \exp\left[-\int_t^T (r(s)+\lambda(s))\,ds\right] + RR\left[\exp\left(-\int_t^T r(s)\,ds\right) - \exp\left[-\int_t^T (r(s)+\lambda(s))\,ds\right]\right].
\end{aligned}$$

Now, if we assume $r(t) = r$ and $\lambda(t) = \lambda$ for all $t \in [0,T]$, the credit spread is given by:

$$\begin{aligned}
CS(t,T) &= -\frac{1}{T-t}\,\log\left[\frac{DB(t,T)}{B(t,T)}\right] \\
&= -\frac{1}{T-t}\,\log\left[e^{-(T-t)\,\lambda} + RR\left(1 - e^{-(T-t)\,\lambda}\right)\right] \\
&= \lambda - \frac{1}{T-t}\,\log\left[1 + RR\left(e^{(T-t)\,\lambda} - 1\right)\right].
\end{aligned}$$

Exercise N° 11.

i) The price of the defaultable ZCB at the beginning of the period is:

$$\begin{aligned} DB(0,T) &= E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(s)\,ds\right)\,\mathbb{I}_{\{\tau>T\}} + \exp\left(-\int_0^\tau r(s)\,ds\right)\,RR(\tau)\,\times\,\mathbb{I}_{\{\tau\le T\}}\right] \\ &= \exp\left(-\int_0^T r(s)\,ds\right)\,E_0^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau>T\}}\right] + \int_0^T e^{-\int_0^s r(u)\,du}\,RR(s)\,d\,F^*(s) \\ &= \exp\left(-\int_0^T r(s)\,ds\right)\,\mathbb{Q}(\tau>T) + \int_0^T e^{-\int_0^s r(u)\,du}\,RR(s)\,d\,F^*(s) \\ &= \exp\left(-\int_0^T r(s)\,ds\right)\,G^*(T) - \int_0^T e^{-\int_0^s r(u)\,du}\,RR(s)\,d\,G^*(s) \end{aligned}$$

where $G^*(t) = 1 - F^*(t) = \mathbb{Q}(\tau > t)$ is the risk-neutral survival probability.

ii) The price of the defaultable ZCB at $t \in (0,T)$, conditionally to $\tau > t$, is:

$$\begin{aligned} DB(t,T) &= E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)\,\mathbb{I}_{\{\tau>T\}} + \exp\left(-\int_t^\tau r(s)\,ds\right)\,RR(\tau)\,\times\,\mathbb{I}_{\{\tau\le T\}}\,|\,\tau>t\right] \\ &= \exp\left(-\int_t^T r(s)\,ds\right)\,\mathbb{Q}(\tau>T\,|\,\tau>t) + \frac{1}{\mathbb{Q}(\tau>t)}\int_t^T e^{-\int_t^s r(u)\,du}\,RR(s)\,d\,F^*(s) \\ &= \exp\left(-\int_t^T r(s)\,ds\right)\,\frac{\mathbb{Q}(\tau>T)}{\mathbb{Q}(\tau>t)} + \frac{1}{\mathbb{Q}(\tau>t)}\int_t^T e^{-\int_t^s r(u)\,du}\,RR(s)\,d\,F^*(s)\,. \end{aligned}$$

This means that $e^{-\int_0^t r(u)\,du}\,DB(t,T)$ can be written in the following:

$$e^{-\int_0^t r(u)\,du}\,DB(t,T) = \exp\left(-\int_0^T r(s)\,ds\right)\,\frac{\mathbb{Q}(\tau>T)}{\mathbb{Q}(\tau>t)} + \frac{1}{\mathbb{Q}(\tau>t)}\int_t^T e^{-\int_0^s r(u)\,du}\,RR(s)\,d\,F^*(s)\,,$$

that is:

$$e^{-\int_0^t r(u)\,du}\,G^*(t)\,DB(t,T) = \exp\left(-\int_0^T r(s)\,ds\right)\,G^*(T) - \int_t^T e^{-\int_0^s r(u)\,du}\,RR(s)\,d\,G^*(s)\,,$$

iii) Contrary to the case in which the recovery is paid at maturity, in this setting if $\tau < t$, we have $\widetilde{DB}(t,T) = 0$ and therefore we can write:

$$\widetilde{DB}(t,T) \;=\; \mathbb{I}_{\{\tau>t\}}\, DB(t,T)\,.$$

Exercise N° 12 [Defaultable ZCB pricing formula in discrete-time with zero recovery].

We consider, in a no-arbitrage discrete-time setting, the problem to price at date t a defaultable ZCB, with maturity date at $t+h$, and issued by the firm i. Let us prove the result recursively on the basis of the following steps:

- Let us start with the case $h = 1$:

$$\begin{aligned}
DB_i(t,t+1) &= E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,|\,I_t] \\
&= E[E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,|\,x_{t+1}, x^i_{t+1}, I_t]\,|\,I_t] \\
&= E[M_{t,t+1}\,E[\mathbb{I}_{\tau_i>t+1}\,|\,x_{t+1}, x^i_{t+1}, I_t]\,|\,I_t] \\
&= E[M_{t,t+1}\,E[\mathbb{I}_{\tau_i>t+1}\,|\,x_{t+1}, x^i_{t+1}, \tau>t]\,|\,I_t] \\
&= E[M_{t,t+1}\,\mathbb{P}(\tau_i>t+1\,|\,x_{t+1}, x^i_{t+1}, \tau>t)\,|\,I_t] \\
&= E[M_{t,t+1}\,\exp(-\lambda^i_{t+1})\,|\,I_t]\,.
\end{aligned}$$

- If we consider the consider the case $h = 2$, we can write:

$$\begin{aligned}
DB_i(t,t+2) &= E[M_{t,t+1}\,M_{t+1,t+2}\,\mathbb{I}_{\tau_i>t+2}\,|\,I_t] \\
&= E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,M_{t+1,t+2}\,\mathbb{I}_{\tau_i>t+2}\,|\,I_t]
\end{aligned}$$

  given that $\mathbb{I}_{\tau_i>t+2} = \mathbb{I}_{\tau_i>t+2}\,\mathbb{I}_{\tau_i>t+1}$. Now, the pricing relationship can be written in the following way:

$$\begin{aligned}
DB_i(t,t+2) &= E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,M_{t+1,t+2}\,\mathbb{I}_{\tau_i>t+2}\,|\,I_t] \\
&= E[E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,M_{t+1,t+2}\,\mathbb{I}_{\tau_i>t+2}\,|\,I_{t+1}]\,|\,I_t] \\
&= E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,E[M_{t+1,t+2}\,\mathbb{I}_{\tau_i>t+2}\,|\,I_{t+1}]\,|\,I_t] \\
&= E[M_{t,t+1}\,\mathbb{I}_{\tau_i>t+1}\,DB(t+1,t+2)\,|\,I_t] \\
&= E[M_{t,t+1}\,DB(t+1,t+2)\,E[\mathbb{I}_{\tau_i>t+1}\,|\,x_{t+1}, x^i_{t+1}, I_t]\,|\,I_t] \\
&= E[M_{t,t+1}\,DB(t+1,t+2)\,\mathbb{P}(\tau_i>t+1\,|\,x_{t+1}, x^i_{t+1}, \tau>t)\,|\,I_t] \\
&= E[M_{t,t+1}\,DB(t+1,t+2)\,\exp(-\lambda^i_{t+1})\,|\,I_t]\,,
\end{aligned}$$

and given that $DB(t+1,t+2) = E[M_{t+1,t+2}\,\exp(-\lambda^i_{t+2})\,|\,I_{t+1}]$ we obtain:

$$\begin{aligned} DB_i(t,t+2) &= E[M_{t,t+1}\,\exp(-\lambda^i_{t+1})\,E[M_{t+1,t+2}\,\exp(-\lambda^i_{t+2})\,|\,I_{t+1}]\,|\,I_t]\,, \\ &= E[M_{t,t+1}\,M_{t+1,t+2}\,\exp(-\lambda^i_{t+1}-\lambda^i_{t+2})\,|\,I_t]\,. \end{aligned}$$

- In the general case, given that $\mathbb{I}_{\tau_i>t+h} = \prod_{j=1}^{h} \mathbb{I}_{\tau_i>t+j}$, we have:

$$\begin{aligned}
& DB_i(t,t+h) \\
= \;& E[M_{t,t+h}\,\mathbb{I}_{\tau_i>t+h}\,|\,I_t] = E\left[M_{t,t+h}\,\prod_{j=1}^{h}\mathbb{I}_{\tau_i>t+j}\,|\,I_t\right] \\
= \;& E\left[M_{t,t+h-1}\,\prod_{j=1}^{h-1}\mathbb{I}_{\tau_i>t+j}\,M_{t+h-1,t+h}\,\mathbb{I}_{\tau_i>t+h}\,|\,I_t\right] \\
= \;& E\left[M_{t,t+h-1}\,\prod_{j=1}^{h-1}\mathbb{I}_{\tau_i>t+j}\,E[M_{t+h-1,t+h}\,\mathbb{I}_{\tau_i>t+h}\,|\,I_{t+h-1}]\,|\,I_t\right] \\
= \;& E\left[M_{t,t+h-1}\,\prod_{j=1}^{h-1}\mathbb{I}_{\tau_i>t+j}\,E[M_{t+h-1,t+h}\,e^{-\lambda^i_{t+h}}\,|\,I_{t+h-1}]\,|\,I_t\right] \\
= \;& E\left[M_{t,t+h-2}\,\prod_{j=1}^{h-2}\mathbb{I}_{\tau_i>t+j}\,E[M_{t+h-1,t+h}\,e^{-\lambda^i_{t+h}}\,|\,I_{t+h-1}]\,M_{t+h-2,t+h-1}\,\mathbb{I}_{\tau_i>t+h-1}\,|\,I_t\right] \\
= \;& E\left[M_{t,t+h-2}\,\prod_{j=1}^{h-2}\mathbb{I}_{\tau_i>t+j}\,E[M_{t+h-1,t+h}\,e^{-\lambda^i_{t+h}}\,|\,I_{t+h-1}]\,E[M_{t+h-2,t+h-1}\,\mathbb{I}_{\tau_i>t+h-1}\,|\,I_{t+h-2}]\,|\,I_t\right] \\
= \;& E\left[M_{t,t+h-2}\,\prod_{j=1}^{h-2}\mathbb{I}_{\tau_i>t+j}\,E[M_{t+h-1,t+h}\,e^{-\lambda^i_{t+h}}\,|\,I_{t+h-1}]\,E[M_{t+h-2,t+h-1}\,e^{-\lambda^i_{t+h-1}}\,|\,I_{t+h-2}]\,|\,I_t\right] \\
\vdots\; & \\
= \;& E_t\Big[M_{t,t+1}\,e^{-\lambda^i_{t+1}}\,E_{t+1}[M_{t+1,t+2}\,e^{-\lambda^i_{t+2}}]\,\cdots \\
& \qquad E_{t+h-2}[M_{t+h-2,t+h-1}\,e^{-\lambda^i_{t+h-1}}]\,E_{t+h-1}[M_{t+h-1,t+h}\,e^{-\lambda^i_{t+h}}]\Big] \\
= \;& E_t[M_{t,t+1}...M_{t+h-1,h}\,\exp(-\lambda^i_{t+1}-...-\lambda^i_{t+h})]\,, \text{ and the result is proved}\,.
\end{aligned}$$