Fixed Income and Credit Risk

Lecture 6

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Lecture 6 - Part I

Interest Rate Derivatives

Outline of Lecture 6 - Part I *(mainly from Veronesi (2010), Chapter 2 and 5)*

6.6 Interest Rate Swaps

6.6.1 Floating Rate Bonds

6.6.2 The Pricing of Floating Rate Bonds

6.6.3 Interest Rate Swaps: Definition

6.6.4 The Value of a Swap

6.6.5 The Swap Rate and the Swap Curve

6.6.6 The LIBOR Yield Curve and the Swap Spread

6.6.7 The Forward Swap Contract and the Forward Swap Rate

6.7 Futures Contracts and Futures Prices

6.8 Futures on Bonds

6.1 Introduction

- **Interest Rate "Derivatives"** play a relevant role in modern financial markets.

- We identify a derivative security as a financial asset whose payoff (and, thus, the price) **depends on the value of some other more basic asset**. That is, the value of the derivative security *DERIVES* from the one of a primitive security.

- That's the traditional meaning, provided by the financial literature in the $'70s$ and the $'80s$ (**Black-Scholes-Merton**), on the basis of the existing basic derivatives, like forwards, futures and swaps. And this what we are going to see in the following sections.

- Nevertheless, today the size of the interest rate derivatives market is much larger than the market size of the primary securities.

- For instance, at the end of 2008, the market size of U.S. Treasury securities was around \$5.9 trillion, while the global market of swaps was about \$16 trillion.

- Given these "numbers", the natural question that stands out is whether the value (the price!) of swaps *depends on* (derives) the value of Treasuries or vice versa.

- In the following sections we will learn that some relations must exist between the value of basic and derivative securities, once the No-Arbitrage principle is applied.

- If these No-Arbitrage-based relations are not respected, then the market (investors/speculators) will immediately take profit of such an opportunity, thus cleaning the market imbalance.

- In other words, we have to figure out that all these markets move jointly and if one moves because of its own **sources of variability (factors!)**, then the others (linked by AAO relationships) should adjust accordingly.

6.2 Forward Rates and Forward Discount Factors

☐ In Lecture 1 we have seen that the annually compounded spot rate $Y(t,T)$ set at date t concerns the price on a loan between the same date t (trading and settlement dates coincide) and the maturity date T.

☐ In the case of **forward rates**, the loan is received at some future settlement date $\tau \geq t$ and the maturity date is (as usual) $T > \tau \geq t$ ($t =$ trading date, $\tau =$ settlement date and $T =$ maturity date).

☐ In other words, this is the rate which is appropriate at time t for discounting between τ and T.

□ The **annually compounded forward rate**, denoted $Y(t, \tau, T)$, with $t \leq \tau < T$, is the rate such that:

$$(1 + Y(t,T))^{-(T-t)} = (1 + Y(t,\tau))^{-(\tau-t)} \times (1 + Y(t,\tau,T))^{-(T-\tau)}\,, \qquad (1)$$

(when $t = \tau$ the forward rate reduces to the spot rate).

□ we can also write (1) in terms of **forward discount factor** $F(t, \tau, T)$:

$$F(t,\tau,T) = \frac{B(t,T)}{B(t,\tau)} = \frac{1}{(1 + Y(t,\tau,T))^{(T-\tau)}}\,. \qquad (2)$$

□ The **forward discount factor** at time t defines the time value of money between two future dates, τ and $T > \tau$, and it is given by the ratio of the two date-t discount factors $B(t, \tau)$ and $B(t, T)$.

☐ The forward discount factor has the following properties: *i*) $F(t, \tau, T) = 1$ for $T = \tau$; *ii*) $F(t, \tau, T)$ is decreasing in T.

☐ The **forward rate** at time t for a risk-free investment from τ to T, and **with compounding frequency** m, is the interest rate determined by $F(t, \tau, T)$:

$$Y^{(m)}(t, \tau, T) = m \times \left(\frac{1}{F(t, \tau, T)^{\frac{1}{m\,(T-\tau)}}} - 1 \right) .$$

☐ The **continuously compounded forward rate** is obtained for $m \to +\infty$:

$$R(t, \tau, T) = -\frac{1}{(T-\tau)} \ln(F(t, \tau, T)) .$$

- Given an m-times compounded forward rate $Y^{(m)}(t,\tau,T)$, the discount factor is:

$$F(t,\tau,T) = \frac{1}{\left(1 + \frac{Y^{(m)}(t,\tau,T)}{m}\right)^{m\,(T-\tau)}}.$$

- Given a continuously compounded forward rate $R(t,\tau,T)$, we have:

$$F(t,\tau,T) = \exp(-\,R(t,\tau,T)\,(T-\tau)).$$

- If the discount factor $B(t,T)$ is increasing between two dates τ and $T>\tau$, that is $B(t,\tau) < B(t,T)$, then the forward rate at t for an investment between τ and T is negative. Nevertheless, as we have seen in Lecture 1, $B(t,T)$ is decreasing in T.

☐ Let us consider the following

- **Example 1:** At date t a firm sold an asset to a client for \$ 100 million. The client will pay in six months from t ($\tau = t + 6$months). Suppose the firm does not need the cash immediately, but it will need it in six months later, at $T = \tau + 6$months.

→ Today, the firm would like to fix the interest rate to be applied on the \$100 million for the six month period $[\tau, T]$. We observe in the bond market that $B(t, \tau) =$ \$97.728 and that $B(t, T) =$ \$95.713 (both with face value 100).

▶ The firm calls up its bank to ask for a quote, and the bank quotes today (at t) the (semi-annually compounded) annualized rate $Y_t^{(2)} = 4.21\%$.

- That is, the bank is ready to commit *today* to receive in six months (at τ) \$100 million from the firm, and return at T the amount:

$$\$102.105\,\text{million} = \$100\,\text{million} \times (1 + Y_t^{(2)}/2)\,.$$

- The rate $Y_t^{(2)} = 4.21\%$ is the *forward rate* $Y^{(2)}(t, \tau, T)$. Indeed, $F(t, \tau, T) = 95.713/97.728 = 0.97938$ and therefore

$$Y^{(2)}(t, \tau, T) = 2 \times \left[\frac{1}{(0.97938)^{1/(2\times 0.5)}} - 1\right] = 4.21\%\,.$$

- Given $B(t, \tau)$ and that $B(t, T)$, it cannot be otherwise than the forward rate without generating an arbitrage opportunity.

Table 1: Bank Trading Strategy to compute forward rate.

Today (date t)	$\tau = t + 6\text{months} = t + 0.5\text{years}$	$T = t + 1\text{year}$
Sell short \$97.728 m of T-bills maturing at τ	a) Receive \$100 m from firm; b) Close short position	
Buy $M = 1.02105 = \frac{\$97.728}{\$95.713}$ m of T-bills maturing at T		a) Receive $1.02105 \times$ \$100 m b) Give total to firm
Total Net CF = 0	Total Net CF = 0	Total Net CF = 0

□ We have also seen during lecture 1, that the **simply compounded forward (LIBOR) rate** at date t, valid for the period $[\tau, T]$, is the rate $L(t,\tau,T)$ such that:

$$B(t,\tau) = B(t,T) \times [1 + L(t,\tau,T) \times (T-\tau)] \,.$$

□ The **simply compounded forward discount factor** $LF(t,\tau,T)$ is thus:

$$LF(t,\tau,T) = \frac{B(t,T)}{B(t,\tau)} = \frac{1}{[1 + L(t,\tau,T) \times (T-\tau)]} \,.$$

□ and therefore

$$L(t,\tau,T) = \frac{1}{T-\tau}\left(\frac{B(t,\tau)}{B(t,T)} - 1\right) \,.$$

□ Let us remember also that the **simply compounded spot (LIBOR) rate** $L(t,T)$ for the period $[t,T]$ is such that the present value of 1 unit of money paid at T is:

$$B_L(t,T) = \frac{1}{[1 + L(t,T) \times (T-t)]} \, ,$$

□ that is:

$$L(t,T) = \frac{1}{T-t} \left(\frac{1}{B_L(t,T)} - 1 \right) \, .$$

6.3 Forward Rate Agreements

□ A **Forward Rate Agreement (FRA)** is a contract between two counterparties, according to which

- one counterparty agrees to pay the (*fixed*) forward rate $Y^{(m)}(t,\tau,T)$ on a given notional amount N during a given a future period of time from τ to T,
- while the other counterparty agrees to pay according to the future market (*floating*) spot rate $Y^{(m)}(\tau,T)$. The net payment between the two counterparties at the maturity T of the contract is given by:

$$\text{"Net payment at } T\text{"} = N \times (T-\tau) \times \left[Y^{(m)}(\tau,T) - Y^{(m)}(t,\tau,T)\right]$$

☐ In the above definition of a FRA, $(T-\tau)$ is typically a quarter or six months, while $m = 1/(T-\tau)$ denotes the corresponding compounding frequency, i.e. $m = 4$ or $m = 2$ for quarterly or semi-annual compounding.

☐ To understand the logic of $FRAs$, let us consider the following example, which is based on the one presented above.

- **Example 2 [Example 1, continued]:** An alternative strategy, to the one described in Example 1, is for the firm to enter into a 6-month FRA with the bank for the period $[\tau, T]$, and notional $N = \$100$ million.

▶ That is, today (at t) the bank agrees to pay in 1 year $(T - t)$ the amount $N \times Y^{(2)}(t, \tau, T)$, while the firm agrees to pay on the same day the amount $N \times Y^{(2)}(\tau, T)$, where $Y^{(2)}(\tau, T)$ is the semi-annually compounded spot interest rate at time τ. That is, they exchange the payment at T:

$$\text{Net payment (payoff) of the firm at } T = \frac{N}{2} \times [Y^{(2)}(t, \tau, T) - Y^{(2)}(\tau, T)]\,;$$

remember that $Y^{(2)}(t, \tau, T) = 4.21\%$.

More precisely: on the firm side...

- At $\tau = t + 6$ months, when the firm receives its \$100 million, the firm can simply invest this amount at the market interest rate $Y^{(2)}(\tau, T)$. How much money will the firm have at time $T = t + 1$ year ? At this time the firm receives the payoff from the investment, plus the net payoff from the FRA. In total:

$$\begin{aligned}\text{Total amount at } T &= \$100\,\text{m} \times \left(1 + \frac{Y^{(2)}(\tau, T)}{2}\right) \quad \text{(Return on investment)} \\ &\quad + \frac{\$100\,\text{m}}{2} \times \left[Y^{(2)}(t, \tau, T) - Y^{(2)}(\tau, T)\right] \quad (FRA \text{ payment}) \\ &= \$102.105\,\text{m}\end{aligned}$$

- The firm is exactly in the same position as Example 1.

... and on the bank side

- The bank now is exposed to interest rate risk, as the FRA yields a negative payoff if $Y^{(2)}(t,\tau,T) > Y^{(2)}(\tau,T)$. Nevertheless, a modification to the (no-arbitrage) trading strategy behind Example 1 (the one justifying the forward rate value!) leads the bank to be hedged. **The new strategy is the following** $\rightarrow$

- at time t the strategy remains the same: the bank shorts \$100 million of 6-month T-bills quoted at $B(t,\tau) = B(t,t+6m) =$ \$97.728 (the market also quotes $B(t,T) = B(t,t+1y) =$ \$95.713) and purchase an amount $M = B(t,\tau)/B(t,T) =$ 1.02105 of $B(t,T)$.

- At time τ the bank must come up with \$100 million to pay the short position. The bank can borrow this amount of money, at the current rate $Y^{(2)}(\tau, T)$. At time T, the bank total cash flow (CF) are:

$$
\begin{aligned}
\text{Total bank CF at } T &= -\$100\,\text{m} \times \left(1 + \frac{Y^{(2)}(\tau, T)}{2}\right) \quad \text{(Pay back loan)} \\
&\quad + \{1.02105 \times \$100\,\text{m}\} \quad \text{(date-}T\text{ T-bills mature)} \\
&\quad - \frac{\$100\,\text{m}}{2} \times [Y^{(2)}(t, \tau, T) - Y^{(2)}(\tau, T)] \quad (FRA \text{ payment}) \\
&= \$0\,,
\end{aligned}
$$

A perfect hedge!

Table 2: Bank Trading Strategy with FRA.

Today (date t)	$\tau = t + 6\text{months} = t + 0.5\text{years}$	$T = t + 1\text{year}$
Sell short \$97.728 m of T-bills maturing at τ	a) Borrow \$100 m at $Y^{(2)}(\tau, T)$ in order to b) close short position	Pay $\$100m \times \left(1 + \frac{Y^{(2)}(\tau,T)}{2}\right)$
Buy $M = 1.02105 = \frac{\$97.728}{\$95.713}$ m of T-bills maturing at T		Receive $1.02105 \times \$100$ m
Enter FRA with firm		Pay $\frac{\$100m}{2} \times [Y^{(2)}(t,\tau,T) - Y^{(2)}(\tau,T)]$
Total Net CF = 0	Total Net CF = 0	Total Net CF = 0

6.3.1 The Value of a Forward Rate Agreement

☐ When the two counterparties enter into a FRA, there is no exchange of money at the time of the contract inception at t.

☐ In other words, the value of the FRA at t is zero!

☐ Nevertheless, as time passes and forward rates change, the value of the FRA changes as well. The following example handle this issue.

- **Example 3 [Example 2, continued]:** Let us imagine that 3 months after the initiation of the contract, that is at date $s = t + 3m \in]\,t, \tau[$, the firm decides to close its FRA with the bank.

▶ At that moment (at s) the two counterparties have not exchanged any money, so it may appear that the firm could simply ask the bank to close the contract. Nevertheless, as interest rates changed between t and s, so did the value of the FRA ($N = 1$ for ease of exposition).

▶ We observe from **Table 2** that at date t the bank sold $N = 1$ T-bill maturing at τ and bought $M = 1.02105$ T-bills maturing at T. This **portfolio** (long on $B(t, T)$ and short on $B(t, \tau)$) exactly hedges the bank commitment to the FRA.

- Since this portfolio produces a cash flow that exactly hedges the one the firm will receive, then the value of this portfolio must reflect the value of the FRA for the firm.

- Thus, for every $s \leq \tau$, we have:

$$\text{Value of } FRA \text{ to the firm at } s \;=\; V^{FRA}(s) = M \times B(s,T) - B(s,\tau)$$

where $M = \dfrac{B(t,\tau)}{B(t,T)}$. For instance, at date t we have $V^{FRA}(t) = M \times B(t,T) - B(t,\tau) = 0$ (no exchange of money at initiation).

▶ At date s, let us imagine that we have $B(s,\tau) = \$99.10$ and $B(s,T) = \$97.37$.

Thus:

$$V^{FRA}(s) = 1.02105 \times B(s,T) - B(s,\tau) = \$0.319638 \neq 0\,.$$

This means that the value of the FRA at s is not zero as at date t.

□ More generally, we can compute the value of the FRA by considering separately the payments of the two counterparties. In particular, let's first decompose the payment of the FRA as follows:

$$\begin{aligned}
\text{Net payment at } T &= N \times (T-\tau) \times \left[Y^{(m)}(t,\tau,T) - Y^{(m)}(\tau,T)\right] \\
&= N \times [1 + Y^{(m)}(t,\tau,T)\,(T-\tau)] - N \times [1 + Y^{(m)}(\tau,T)\,(T-\tau)] \\
&= \textbf{Fixed leg payment - Floating leg payment}\,.
\end{aligned}$$

□ Let us now value the two legs separately. The **fixed leg** is the simplest one, given that it corresponds to a fixed payment at T and thus we can discount it, as if it was a ZCB:

$$\begin{aligned} V^{fixed}(s) &= \text{Present value of}\, N \times [1 + Y^{(m)}(t,\tau,T)\,(T-\tau)] \\ &= B(s,T) \times N \times [1 + Y^{(m)}(t,\tau,T)\,(T-\tau)]. \end{aligned}$$

□ With regard to the value of the **floating leg**, observe that at date s we do not know the future $Y^{(m)}(\tau,T)$. We handle this problem following a **two-step procedure**. First, we compute the value of the floating leg at τ:

$$\begin{aligned} V^{float}(\tau) &= \text{Present value of}\, N \times [1 + Y^{(m)}(\tau,T)\,(T-\tau)] \\ &= B(\tau,T) \times N \times [1 + Y^{(m)}(\tau,T)\,(T-\tau)] \\ &= \frac{1}{[1 + Y^{(m)}(\tau,T)\,(T-\tau)]} \times N \times [1 + Y^{(m)}(\tau,T)\,(T-\tau)] = N\,. \end{aligned}$$

- We observe that $V^{float}(\tau) = N$ *independently* of the floating rate $Y^{(m)}(\tau, T)$, i.e. the date-τ value is deterministic. Thus, we can discount $V^{float}(\tau)$ from τ to s by means of $B(s, \tau)$:

$$V^{float}(s) = B(s, \tau) \times N\,.$$

- Combining $V^{fixed}(s)$ and $V^{float}(s)$ we have that the **value** at time s of the FRA, in which the counterparties exchange at T the cash flow:

$$\text{Net payment at } T = N \times (T - \tau) \times \left[Y^{(m)}(t, \tau, T) - Y^{(m)}(\tau, T)\right] ,$$

is given by

$$\begin{aligned} V^{FRA}(s) &= B(s, T) \times N \times [1 + Y^{(m)}(t, \tau, T)\,(T - \tau)] - B(s, \tau) \times N \\ &= N \times [B(s, T)\,(1 + Y^{(m)}(t, \tau, T)\,(T - \tau)) - B(s, \tau)] \end{aligned}$$

□ Now, given that $1 + Y^{(m)}(t,\tau,T)\,(T-\tau) = \dfrac{B(t,\tau)}{B(t,T)} = M$, we can write:

$$V^{FRA}(s) \;=\; N \times [B(s,T) \times M - B(s,\tau)]\,.$$

□ The definition of M ensures that at the inception date t:

$$V^{FRA}(t) \;=\; N \times [B(t,T) \times M - B(t,\tau)] = 0\,.$$

□ We can equivalently write:

$$\begin{aligned} V^{FRA}(s) &= N \times B(s,T) \times \left[M - \frac{B(s,\tau)}{B(s,T)}\right] \\ &= N \times B(s,T) \times (T-\tau) \times [Y^{(m)}(t,\tau,T) - Y^{(m)}(s,\tau,T)]\,. \end{aligned}$$

6.3.2 A Stochastic Discount Factor approach to Forward Rate Agreements

- ☐ We have seen in the previous slides that a FRA is a contract involving three time instants: $t < \tau < T$, with t the current date and T the maturity date.

- ☐ It is important to realize that, assuming $m = 1/(T - \tau)$ means that $F(t, \tau, T) = LF(t, \tau, T)$ and $Y^{(m)}(t, \tau, T) = L(t, \tau, T)$ (the same for spot rate and factors).

- ☐ The contract gives its holder an interest-rate payment for the period $[\tau, T]$.

□ At maturity T, a **fixed payment** based on a **fixed rate** K (i.e., fixed/known at t) is exchanged against a **floating payment** based on the simply compounded spot rate $L(\tau, T)$.

□ Basically, this contract allows one to lock-in the interest rate between τ and T at a desired value K. Formally, at T one receives $(T - \tau)\, K\, N$ units of currency and pays the amount $(T - \tau)\, L(\tau, T)\, N$.

☐ The date-T payoff is:

$$\begin{aligned} y(T) &= N \times (T-\tau) \times [K - L(\tau,T)] \\ &= N \times [1 + K\,(T-\tau)] - N \times [1 + L(\tau,T)\,(T-\tau)] \\ &= \textbf{Fixed leg payment - Floating leg payment}\,. \end{aligned}$$

☐ What the **fixed rate** K is ?

☐ Given that at the initiation date t there is no cash flow, the value of the FRA contract is zero: $V^{FRA}(t) = 0$.

☐ We will use this condition (and the SDF $M_{t,T}$) to find the fixed rate K such that $V^{FRA}(t) = 0$.

□ We know that, under the A.A.O., we can write:

$$
\begin{aligned}
V^{FRA}(t) &= E_t[M_{t,T}\, y(T)] = E_t\left\{M_{t,T}\, N \times (T-\tau) \times [K - L(\tau,T)]\right\} \\
&= N \times [1 + K\,(T-\tau)]\, E_t[M_{t,T}] - N \times E_t\left\{M_{t,T} \times [1 + L(\tau,T)\,(T-\tau)]\right\} \\
&= N \times [1 + K\,(T-\tau)]\, B(t,T) - N \times E_t\left\{M_{t,T} \times [1 + L(\tau,T)\,(T-\tau)]\right\}
\end{aligned}
$$

□ From $B(\tau,T) = (1 + L(\tau,T)\,(T-\tau))^{-1}$ we have:

$$
\begin{aligned}
V^{FRA}(t) &= N \times [1 + K\,(T-\tau)]\, B(t,T) - N \times E_t\left\{M_{t,\tau}\, M_{\tau,T} \times B^{-1}(\tau,T)\right\} \\
&= N \times [1 + K\,(T-\tau)]\, B(t,T) - N \times E_t\left\{M_{t,\tau}\, E_\tau[M_{\tau,T} \times B^{-1}(\tau,T)]\right\} \\
&= N \times [1 + K\,(T-\tau)]\, B(t,T) - N \times E_t\left\{M_{t,\tau}\right\} \\
&= N \times [1 + K\,(T-\tau)]\, B(t,T) - N \times B(t,\tau)\,.
\end{aligned}
$$

☐ Clearly:

$$V^{FRA}(t) = 0 \quad \Longleftrightarrow \quad 1 + K\,(T - \tau) = \frac{B(t,\tau)}{B(t,T)}$$

$$\Longleftrightarrow \quad K = \frac{1}{(T-\tau)}\left[\frac{B(t,\tau)}{B(t,T)} - 1\right] = L(t,\tau,T)\,.$$

☐ We have that the date-t **fixed rate** K that renders the contract fair at time t is the **simply compounded forward rate** $L(t,\tau,T)$.

☐ Thus, here we observe that forward rates are interest rates that can be locked in today (date t) to hedge interest rate risk affecting an investment over $[\tau, T]$, and are set *consistently* with the current term structure of discount factor.

6.4 Forward Contracts and Forward Prices

- ☐ In a FRA, two counterparties agree to exchange cash flows according to the difference between the fixed (at t) forward rate (known at the initiation of the contract) and the future spot rate.

- ☐ An equivalent strategy for an investor to lock in a given rate of return in the future is to agree to purchase a given Treasury security in the future, at a price determined today.

- ☐ This equivalent strategy consists in entering into a **forward contract** with **underlying asset** a given Treasury security.

☐ Let us first provide the general definition of **forward contract**, and then we will see the particular case in which the underlying is a Treasury Bond.

☐ A **forward contract :**

- is an agreement, signed at the initial date $t = 0$,
- to buy or sell an **asset** or a **specific interest rate** valued V_T at a given future date T, called **delivery date** or **maturity**,
- for a prespecified price K, called the **delivery price**.

☐ Since a forward contract is settled at maturity and a party in the long (respectively, short) position is obliged to buy (respectively, to sell) the asset being worth V_T at maturity for K,

⇒ the **payoff** of the long (respectively, short) position is the **contingent claim** $P_T = V_T - K$ (respectively, $-P_T$).

☐ Observe that there is no cash flow at $t = 0$ and, therefore, the **price of a forward contract at its initial date is zero**.

☐ The **delivery price** K which induces a **worthless forward contract at** $t = 0$ is called the **forward price** (of the underlying asset valued V_T for the settlement date T) and is denoted $\Phi(0, T)$.

☐ For all **initial dates** $t < T$, the forward price will be denoted $\Phi(t, T)$ and, for $t = T$ (immediate delivery), we have by definition $\Phi(T, T) := V_T$.

☐ The forward price at date t is given by

$$\begin{aligned} \Phi(t,T) &= \frac{E_t\left[M_{t,t+1} \cdot \ldots \cdot M_{T-1,T}\, V_T\right]}{B(t,T)} = \frac{E_t^{\mathbb{Q}}\left[\exp(-r_t - \cdot \ldots \cdot - r_{T-1})\, V_T\right]}{B(t,T)} \\ &= \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{T-1} r_i\right), V_T\right]}{B(t,T)} + E_t^{\mathbb{Q}}[V_T] \end{aligned} \tag{3}$$

□ and satisfies the recursive relation :

$$\begin{aligned} \Phi(t,T) &= E_t\left[M_{t,t+1}\,\frac{B(t+1,T)}{B(t,T)}\,\Phi(t+1,T)\right] \\ &= E_t^{\mathbb{Q}}\left[\,\frac{B(t,t+1)\,B(t+1,T)}{B(t,T)}\,\Phi(t+1,T)\right]\ , \end{aligned} \tag{4}$$

where $B(t,T)$ is the price, at date t, of a zero-coupon risk-less bond maturing at time T.

□ **Proof :** Because the price of the forward contract is zero we have, under the no-arbitrage principle, that at any date t :

$$0 = E_t\left\{M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}\ \times\ [V_T - \Phi(t,T)]\right\}\ .$$

□ Solving for the forward price we obtain :

$$\Phi(t,T) = \frac{E_t\left[M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}\,V_T\right]}{B(t,T)} = \frac{E_t\,E_{t+1}\left[M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}\,V_T\right]}{B(t,T)}$$

$$= E_t\left[M_{t,t+1}\frac{B(t+1,T)}{B(t,T)}\,\Phi(t+1,T)\right] .$$

□ **Remark 1 :** Relation (3) implies that **if the short rate process** $(r_t)_{0\le t\le T-1}$ **and** V_T are **uncorrelated under** $\mathbb{Q}$, the forward price process $\{\Phi(t,T)\}_{0\le t\le T-1}$ is a $\mathbb{Q}$-**martingale**, that is, it verifies for every $0 \le t \le T-1$ the following relation :

$$\Phi(t,T) = E_t^{\mathbb{Q}}(V_T) = E_t^{\mathbb{Q}}\left[\Phi(t+1,T)\right] . \tag{5}$$

This would be true, for instance, if the short rate process is known in advance.

□ **Remark 2 :** In the case of absence of dividends, the forward price at date t is given by:

$$\Phi(t,T) = \frac{V_t}{B(t,T)} \,. \tag{6}$$

□ The **price** at t of a **forward contract** signed at 0 :

$$P_t \;=\; [\,\Phi(t,T) \;-\; \Phi(0,T)\,]\; B(t,T)\;. \tag{7}$$

□ **Proof :**

$$\begin{aligned}
P_t &= E_t \left\{ M_{t,t+1} \cdot \ldots \cdot M_{T-1,T} \, [V_T - \Phi(0,T)] \right\} \\
&= E_t \left(M_{t,t+1} \cdot \ldots \cdot M_{T-1,T} \; V_T \right) - \Phi(0,T)\, B(t,T) \\
&= \Phi(t,T)\, B(t,T) - \Phi(0,T)\, B(t,T)\,.
\end{aligned}$$

6.5 Forwards on Bonds

- Let us consider a **forward contract** (with delivery date τ) **on a zero-coupon bond** with maturity date $T \geq \tau$ whose date t price is denoted by $B(t,T)$. Here, since there is no intermediate payoffs, we get:

- The forward price, at date t with delivery date τ, on a zero-coupon bond maturing at date $T \geq \tau$ is

$$\begin{aligned} \Phi(t,\tau,T) &= \frac{E_t^{\mathbb{Q}}\left[\exp(-r_t - \cdot \ldots \cdot - r_{\tau-1})\; B(\tau,T)\right]}{B(t,\tau)} \\ &= \frac{B(t,T)}{B(t,\tau)} = F(t,\tau,T) \;\; \text{(the forward discount factor)}. \end{aligned} \tag{8}$$

□ **The No-Arbitrage Argument :** let us imagine that $\Phi(t, \tau, T) > F(t, \tau, T)$, where $F(t, \tau, T) = \dfrac{B(t, T)}{B(t, \tau)}$.

- At time t an arbitrageur can (with net cash flow = 0):

1. Sell the forward zero coupon $B(\tau, T)$ at forward price $\Phi(t, \tau, T)$.

2. Short exactly $F(t, \tau, T)$ zero coupon bonds with maturity τ for the amount $F(t, \tau, T) \times B(t, \tau) = B(t, T)$, the price of a zero coupon with maturity T.

3. Use the proceeds from 2. to purchase one zero coupon with maturity T.

- At time τ the arbitrageur:

  1. Delivers the zero coupon $B(\tau, T)$, which he/she owns, and receives $\Phi(t, \tau, T)$, thereby closing the forward contract.

  2. Covers the short position paying $F(t, \tau, T)$.

□ The net cash position between t and τ is positive, with no uncertainty or risk in this strategy, thus it is an arbitrage opportunity.

☐ **Example 4:** On March 1, 2001 (today $= t = 0$), the firm may enter into a forward contract with a bank to purchase six months later ($T_1 = 0.5$) \$100 million-worth of 6-months Treasury bills ($T_2 = 1 > T_1$) for a price $\Phi(0, 0.5, 1)$, for \$100 par value, specified today.

- What purchase price would the bank quote to the firm for the 6-month T-bills? Applying formula (8), we find that the no-arbitrage price is:

$$\Phi(0, 0.5, 1) = 100 \times F(0, 0.5, 1) = \$97.938\,.$$

- Recall that from the (short) sale of T_1 T-bills, the bank purchases $M = 1.02105$ million of T_2 T-bills. Recall also that the net cash flow to the bank at time $t = 0$ is zero.

- At time T_1 the bank has to cover the short position, and uses the \$100 million from the firm. Note that at this point the bank holds an amount $M = 1.02105$ million of T_2 T-bills, which now have maturity $T_2 - T_1 =$ six months.

- Therefore, the bank can use these M 6-month T-bills to honor the terms of the forward contract. That is, at T_1 the bank simply delivers its own holdings M of 6-month T-bills to the firm. This number M of T_2 T-bills is exactly the number of 6-month T-bills that are needed to ensure the firm gets \$100 million-worth of 6-month T-bills, as the firm requested; in fact, given $\Phi(0, 0.5, 1)$, we have:

$$\frac{\$100\,\texttt{million}}{\Phi(0, 0.5, 1)} = 1.02105 = M\,.$$

☐ More generally we can consider a forward contract on a coupon bond with payments C_i at time T_i, $i \in \{1, \dots, n\}$. Let us consider the coupon bond price at time $T < T_1$, given by $CB(T, \widetilde{T}) = \sum_{i=1}^{n} C_i \, B(T, T_i)$ (with $T_n = \widetilde{T}$), then the forward price at time t with delivery date T is given by :

$$
\begin{aligned}
\Phi^{CB}(t, T, \widetilde{T}) &= \frac{E_t \left[M_{t,t+1} \cdot \ldots \cdot M_{T-1,T} \; CB(T, \widetilde{T}) \right]}{B(t,T)} \\
&= \frac{\sum_{i=1}^{n} C_i \; E_t \left[M_{t,t+1} \cdot \ldots \cdot M_{T-1,T} \; B(T, T_i) \right]}{B(t,T)} \\
&= \frac{\sum_{i=1}^{n} C_i \; B(t, T_i)}{B(t,T)} = \sum_{i=1}^{n} C_i \; \Phi(t, T, T_i)
\end{aligned}
\tag{9}
$$

where $\Phi(t, T, T_i)$ is the forward price on a ZCB maturing at date T_i.

□ In this case the **value of forward contract** for every $t < T$ is:

$$P_t = [\Phi(t, \tau, T) - \Phi(0, \tau, T)] \times B(t, T) . \tag{10}$$

- where $\Phi(0, \tau, T)$ is the delivery price specified at the initiation of the contract.

□ Example 5: Pricing a Forward on Bonds with Gaussian AR(1) ATSM

- If we consider a Gaussian AR(1) Factor-Based term structure model, then we know that $B(t,T) = \exp[c_{T-t}x_t + d_{T-t}]$ with:

$$\begin{cases} c_{T-t} &= -\alpha + \varphi^* c_{T-t-1}\,, \\ d_{T-t} &= -\beta + c_{T-t-1}\nu^* + \frac{1}{2}c^2_{T-t-1}\sigma^2 + d_{T-t-1}\,, \end{cases}$$

with $\varphi^* = (\varphi + \sigma\gamma)$, $\nu^* = (\nu + \gamma_o\sigma)$.

- Thus, the Forward price $\Phi(t,T,S)$, associated to an underlying ZCB with price $V_T = B(T,S)$ at the delivery date T, is given by:

$$\Phi(t,T,S) = \frac{B(t,S)}{B(t,T)} = \exp[(c_{S-t} - c_{T-t})x_t + (d_{S-t} - d_{T-t})]\,.$$

6.6 Interest Rate Swaps

6.6.1 Floating Rate Bonds

- ☐ In order to properly introduce the concept (and the pricing) of Interest Rate Swaps, we have first to talk about **Floating Rate Bonds** (FRBs).

- ☐ FRBs are coupon bonds whose coupons are tied to some reference interest rate. The U.S. Treasury does not issue FRBs, but government of other countries as well as individual corporations do.

- ☐ For ease of presentation, we only consider the case in which the reference rate coincide with the discounting rate.

□ **Definition:** *A semi-annual* **Floating Rate Bond** *with maturity T is a bond whose coupon payments $C(T_i)$ at dates $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$, ... $T_n = T$ are determined by the formula:*

$$C(T_i) = 100 \times [r_2(T_i - 0.5) + s],$$

where $r_2(t) := R(t, t + 0.5)$ is the 6-month Treasury rate at t, and s is a spread. Each coupon date is also called **reset date** *as it is the time when the new coupon is reset.*

6.6.2 The Pricing of Floating Rate Bonds

☐ Let us start with the case in which the spread $s = 0$. In this case the following is true:

- If $s = 0$, the **ex-coupon price** of a Floating Rate Bond **on any coupon date** is equal to the bond par value (the principal).

☐ **Example 6:** Consider a one year, semi-annual floating rate bond, with zero spread. The coupon at $t = 0.5$ depends on today's interest rate $r_2(0) = 2\%$, then $C(0.5) = 100 \times 2\% / 2 = 1$.

What about the coupon payment $C(1)$ at maturity $T = 1$?

- The last coupon rate $C(1)$ depends on the interest rate in six months $r_2(0.5)$ which is unknown today, yet this doesn't matter since the cash flow at that time will be $(100 + C(1))$, which means that the present value at $t = 0.5$ will be:

$$CB_{FR}(0.5, 1) = \frac{100 \times (1 + r_2(0.5)/2)}{1 + r_2(0.5)/2} = 100\,.$$

→ *independently* of the level of interest rate $r_2(t = 0.5)$, $CB_{FR}(0.5, 1)$ will always be 100. After the coupon is paid at $t = 0.5$, the value of the bond is the face value (\$100), so the value at $t = 0$ is:

$$CB_{FR}(0, 1) = \frac{100 + 1}{1 + 2\%/2} = 100\,.$$

and this is because reference rate and discounting rate coincide.

☐ What if $s \neq 0$ but $t = 0$ is a reset date? Effectively the spread is a fixed payment on the bond so we can value it separately:

$$\texttt{Price of FRB with spread = (Price of FRB with } s=0\texttt{)} + s \times 100 \times \sum_{t=0.5}^{n} B(0,t)$$

$$= 100 + s \times 100 \times \sum_{t=0.5}^{n} B(0,t)\,.$$

☐ How do we value a floating rate bond outside of reset dates?

- We know that the cum-coupon value will be $CB_{FR}^{C}(t_i, T) = 100(1 + r_2(t_i)/2)$ at the next reset date, and note that $r_2(t_i)$ is known.

- All we need to do is to apply the appropriate discount.

- This leads to the following general formula.

☐ Let $T_1, T_2, \ldots, T_n$ be the floating rate reset dates and let the current date t be between time T_i and T_{i+1}: $T_i < t < T_{i+1}$:

- the general formula for a semi-annual floating rate bond with zero spread s is:

$$\begin{aligned} CB_{FR}(t,T) &= \texttt{Present Value of } CB^C_{FR}(T_{i+1},T) \\ &= B(t,T_{i+1}) \times 100 \times [1 + r_2(T_i)/2], \end{aligned}$$

where $B(t,T_{i+1})$ is the discount factor from t to T_{i+1}.

- at reset dates $B(T_i,T_{i+1}) = 1 + r_2(0.5)/2$, which implies $CB_{FR}(t,T) = 100$.

6.6.3 Interest Rate Swaps: Definition

☐ A plain vanilla **fixed-for-floating interest rate swap** contract is an agreement between two counterparties in which

- one counterparty agrees to make ℓ fixed payments per year at an (annualized) rate c on a notional N up to a maturity date T,
- while at the same time the other counterparty commits to make payments linked to a floating (time varying) rate index $r_\ell(t)$.

☐ Let us denote by $T_1, T_2, \ldots, T_n = T$ the payment dates, with $T_i = T_{i-1} + \Delta$ and $\Delta = 1/\ell$. We have in general $\ell = 2$ (semiannual fixed payments).

☐ The net payment between the two counterparties at each of these dates is:

$$\texttt{Net Payment at } T_i = N \times \Delta \times [r_\ell(T_{i-1}) - c]$$

- The constant c is called the **Swap Rate**.

- The reference rate for the payment at time T_i is not the rate at T_i, but the one determined six months before, at T_{i-1}.

- The two counterparties agree to exchange cash flows in the *future*, not *today*. Therefore, there is a zero net cash flow at the initiation date (like for a Forward contract and a FRA).

□ **Example 7:** A firm and bank decide to enter into a fixed for floating, semi-annual, 5-year swap with swap rate $c = 5.46\%$ and notional amount $N = \$200$ million; the reference floating rate is the 6-month LIBOR.

- In this swap contract, the firm agrees to pay to the bank every six months ($T_i = 0.5, 1, 1.5, ..., 5$):

$$\texttt{Cash flow from firm to bank at } T_i = \$200\,m \times 0.5 \times 5.46\% = \$5.46\,m\,.$$

- In exchange, the bank pays the firm at every T_i an amount that depends on the 6-month LIBOR $r_2(T_{i-1})$:

$$\texttt{Cash flow from bank to firm at } T_i = \$200\,m \times 0.5 \times r_2(T_{i-1})\,.$$

- Table 5.4 illustrates the cash flows from the bank to the firm and vice versa. The noteworthy point is that the cash flows from the bank to the firm in Column 3 vary over time, and in particular they have a **six months lag** from the time the LIBOR, in Column 2, is realized.

- In this particular instance, the firm would receive a negative net cash flow, as the reference floating rate declined from 4.951% at initiation to a much lower number.

Table 5.4 Example Cash Flows in Fixed-for-Floating Swap

Time	LIBOR	Flow from Bank to Firm	Flow from Firm to Bank	Net Cash Flow to Firm
0.0	4.951 %			
0.5	3.460 %	$ 4.951 m	$ 5.460 m	$ -0.509 m
1.0	2.040 %	$ 3.460 m	$ 5.460 m	$ -2.000 m
1.5	1.800 %	$ 2.040 m	$ 5.460 m	$ -3.420 m
2.0	1.339 %	$ 1.800 m	$ 5.460 m	$ -3.660 m
2.5	1.201 %	$ 1.339 m	$ 5.460 m	$ -4.121 m
3.0	1.170 %	$ 1.201 m	$ 5.460 m	$ -4.259 m
3.5	1.980 %	$ 1.170 m	$ 5.460 m	$ -4.290 m
4.0	3.190 %	$ 1.980 m	$ 5.460 m	$ -3.480 m
4.5	3.996 %	$ 3.190 m	$ 5.460 m	$ -2.270 m
5.0	4.976 %	$ 3.996 m	$ 5.460 m	$ -1.464 m

☐ **Example 8:** Today is $t = 0 =$ March 1, 2001, consider a firm that sold a piece of equipment to a highly rated corporation, and it is then due to receive payments in 10 equal installments of \$5.5 million each over 5 years.

- The firm would like to use these \$5.5 million semi-annual cash flows to hedge against the coupon payments the firm must make to service a \$200 million floating rate bond that it issued in the past, and also expiring in 5 years.

- Suppose that the floating rate on the corporate bond is tied to the LIBOR, at LIBOR $+ 4\ bps$.

- The 6-month LIBOR on March 1, 2001 is currently at 4.95% and so the next interest rate payment the firm must make is $(4.95 + 0.04)\%/2 \times 200$ million = \$4.9 million; thus, the next floating rate coupon payment is covered.

- However, if the LIBOR were to increase by more than 0.51% in the next 5 years, the cash flows from the installments would not be sufficient to service the debt.

- A solution is to enter into a fixed-for-floating swap with an investment bank, in which the firm pays the fixed semi-annual swap rate c, over a notional of \$200 million, and the bank pays the 6-month LIBOR to the firm.

- On March 1, 2001, the swap rate for a 5-year fixed-for-floating swap was quoted at $c = 5.46\%$. So, in this case, the net cash flow to the firm from the swap contract is:

$$\texttt{Net Payment at}\, T_i = \$200\, m \times 0.5 \times [r_2(T_{i-1}) - 5.46\%]$$

where $r_2(t)$ is the six month LIBOR at time t.

- Why does this swap resolve the problem?

- Consider the net position of the firm at every T_i:
  - receives 5.5 million
  - pays $(r_2(T_i - 0.5) + 4bps)/2 \times 200$ million on its outstanding floating rate debt
  - receives $r_2(T_i - 0.5)/2 \times 200$ million from the bank as part of the swap
  - pays 5.46% $\times$ 0.5 $\times$ 200 million to the bank as part of the swap

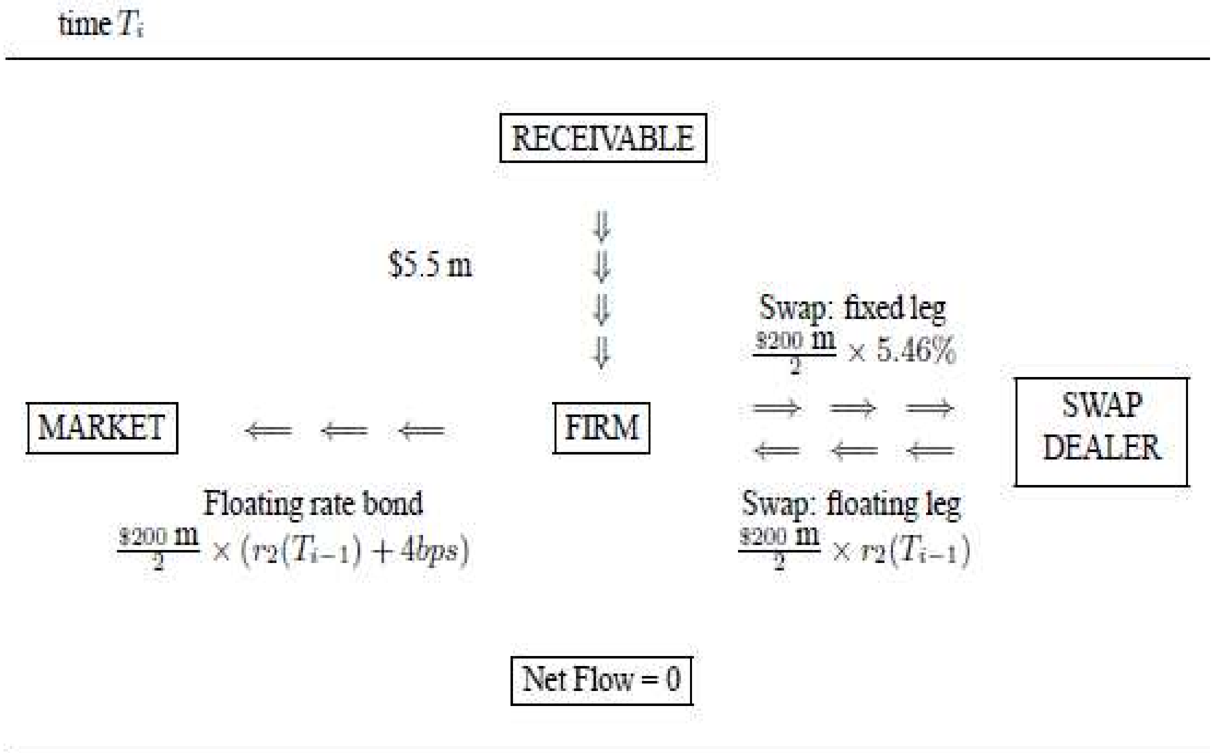
- Summing up, the firms net cash flow position from the receivable, debt, and swap is:

$$\begin{aligned} & \texttt{Total Cash Flow at}\, T_i \\ = \ & \$5.5\, m \ \texttt{(Receivable)} \\ & -(r_2(Ti - 0.5) + 4bps)/2 \times 200m \ \texttt{(Debt)} \\ & +0.5 \times [r_2(T_i - 0.5) - 5.46\%] \times 200m \ \texttt{(Swap)} \\ = \ & 5.5 - 0.04\% \times 100 - 5.46\% \times 100 = 0 \end{aligned}$$

- That is, the firm is perfectly hedged: The risk in the fluctuations of the LIBOR stemming from its liabilities has been eliminated by the swap (the firm receives the LIBOR from the bank, and pays the LIBOR + 0.04% to bond holders).

- The remaining fixed components sum up to zero.

Figure 5.3 Hedging with Swaps



6.6.4 The Value of a Swap

☐ How do we value a swap?

☐ The sequence of net cash flows is the same as the one of a portfolio that is:

- long a floating rate bond, and
- short a fixed rate bond with coupon c

$$V_{swap}(t,T;c) = CB_{FR}(t,T) - CB_c(t,T)$$

where $V_{swap}(t,T;c)$ denotes the values of the swap at time t, with swap rate c and maturity date $T_n = T$; $CB_{FR}(t,T)$ is the value of the floating rate bond and $CB_c(t,T)$ is the value if the coupon bond with fixed coupon rate c.

☐ At the payment dates T_i, we have $CB_{FR}(t,T) = 100$ and so:

$$V_{swap}(T_i, T; c) = 100 - \left[\sum_{j=i+1}^{n} \frac{c}{2} \times 100 \times B(T_i, T_j) + 100 \times B(T_i, T_n) \right]$$

☐ How is the swap rate c determined ? The contract specification implies that there is no exchange of money at the inception of the contract.

☐ This means that the value of the contract at the inception is zero, and c is determined accordingly.

6.6.5 The Swap Rate and the Swap Curve

☐ The **Swap Rate** c is given by that number that makes the value of the swap at the initiation date equal to zero, namely $V_{swap}(0,T;c) = 0$.

☐ Rewriting the swap value equation (generically) for any number n of payment taking place at dates $T_1, T_2, \ldots, T_n$:

$$V_{swap}(0,T;c) = 100 - \left[\sum_{j=1}^{n} \frac{c}{2} \times 100 \times B(0,T_j) + 100 \times B(0,T_n) \right]$$

and solving for c we have:

$$c = 2 \times \frac{1 - B(0,T_n)}{\sum_{j=1}^{n} B(0,T_j)}$$

☐ You may easily verify that, in the previous example, $c = 5.46\%$ makes the swap value of the inception of the contract equal to zero. In other words, $c = 5.46\%$ is the proper swap rate.

☐ Now, what are the appropriate discount factors $B(t, T)$, to price swaps, given that, over the years, the swap market grew so much that market forces determine the swap rate for every possible future maturity? Let us first introduce the notion of swap curve.

☐ **Definition:** *The* **Swap Curve** *at time t is the set of swap rates (at time t) for all maturities* T_1*,* T_2*, ...,* $T_n = T$*. We denote the swap curve at t by* $c(t, T_i)$ *for* $i = 1, \ldots, n$.

- Given the size of the swap market [$8 trillion, against $87 billions of FRA and $1.1 trillion of OTC options as of December 2008 (source: BIS); the size of Treasury debt at that time was $5.9 trillion], the swap curve $c(t,T)$ has become in fact a reference point to determine the time value of money for financial institutions.

- Indeed, given $c(t,T_i)$, we can compute the implicit discount factors $B(t,T_i)$ by applying a bootstrap methodology similar to the one discussed during Lecture 2. Specifically, we can invert the swap rate formula and find for $i=1$:

$$B(t,T_1) = \frac{1}{1+\frac{c(t,T_1)}{2}},$$

- while for $i = 2, \dots, n$

$$B(t,T_i) = \frac{1 - \dfrac{c(t,T_i)}{2} \times \displaystyle\sum_{j=1}^{i-1} B(t,T_j)}{1 + \dfrac{c(t,T_i)}{2}}.$$

☐ **Example 9:** Let us imagine at the date $t = 0$, the swap rates, for maturities $T_1 = 0.5$, $T_2 = 1$ and $T_3 = 1.5$, are respectively given by $c(0,T_1) = 4.951\%$, $c(0,T_2) = 4.910\%$ and $c(0,T_3) = 4.980\%$. Applying the above presented formula, the associated discount factors are $B(0,0.5) = 0.9758$, $B(0,1) = 0.9527$ and $B(0,1.5) = 0.9289$.

6.6.6 The LIBOR Yield Curve and the Swap Spread

- ☐ Let us denote the discount factors extracted from the swap rates by $B^L(t,T)$, where the superscript "L" stands for LIBOR.

- ☐ Indeed, it is customary to refer to the swap-rate implied discount factors as the LIBOR discount, and its term structure as the **LIBOR Yield curve**. The reason is that the underlying floating rate is the LIBOR.

- ☐ What is the relationship between the LIBOR Yield curve and the one obtained from Treasuries ?

- The difference between LIBOR and Treasury yields with same maturities, is called the **swap spread**, and it is generally assumed that it is unlikely to become negative given that probability of default by swap dealers is higher than the governments.

- The spread is also very small, in general, but it can become quite substantial during turbulent periods, such as the credit crisis of 2007-2008.

Figure 5.4 Treasury and Swap Discount and Yield on January 4, 2005

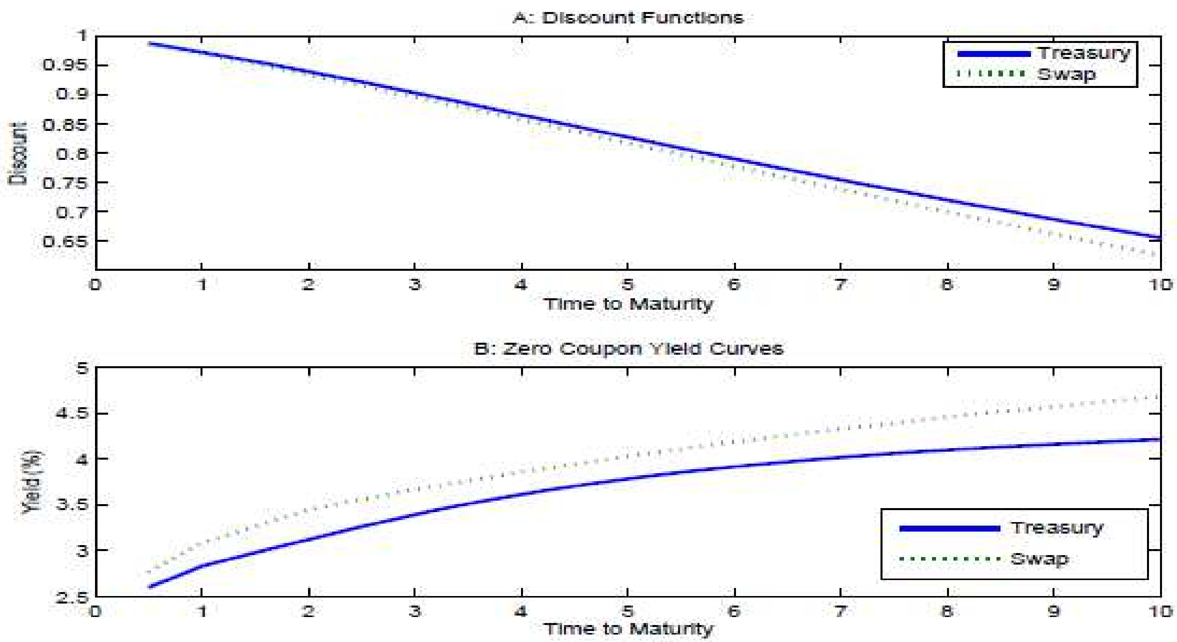
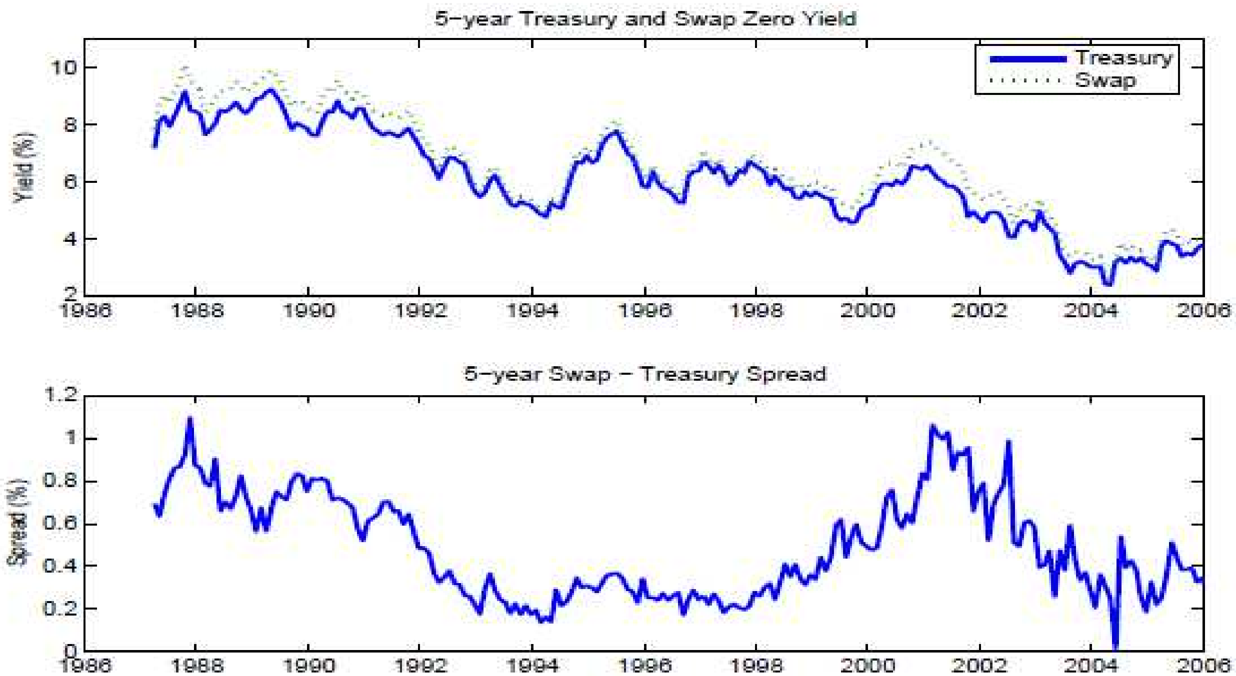


Figure 5.5 The 5-year Zero Coupon Treasury and Swap Yield



6.6.7 The Forward Swap Contract and the Forward Swap Rate

- □ The same way it is possible to lock in a future interest rate today by entering into a FRA, it is also possible to lock in a future swap rate by entering into a Forward Swap contract.

- □ **Definition:** *The* **Forward Swap contract** *is a contract in which two counter-parties agree to enter into a swap contract at a predetermined future date and for a predetermined swap rate* f^s*, called the* **forward swap rate**.

☐ **Example 10:** Let us consider again Example 8 but assume now that on March 1, 2001, the firm signed a contract to deliver one year later (on March 1, 2002) a large piece of equipment. The payment will be made in 8 equal installments of $5.5 million each over the next 4 years, starting on September 1, 2003.

- Assume the firm plans to use these cash inflows to meet the payments of a floating rate bond issued some time in the past. As explained in the previous example, the firm can enter into a fixed-for-floating swap in which it pays fixed and receives floating.

- The problem is that the firm will start receiving payments much further in the future, and therefore it will need to enter into such a fixed-for-floating swap one year from now, on March 1, 2002.

- The firm is worried however that the 4-year swap rate may increase between now and March 1, 2002, an event that may unduly increase its cash outflows from the hedging program.

- Therefore, the firm decides to enter into a forward contract with a bank, in which the bank and the firm agree *today* that the 4-year swap rate in the future will be $f_2^s = 5.616\%$, to be paid semi-annually in exchange of the 6-month LIBOR.

☐ The question then is the usual: How can the bank commit today to enter into a swap contract in the future at a given swap rate f_2^s ?

☐ To answer this question, we may recall that the value of a swap [in which the counterparty receives the fixed rate c] can be seen as a portfolio that is (see Section 6.6.3):

- short a floating rate bond (with value of 100 at reset dates)
- and long a fixed rate bond with coupon rate c.

□ The payoff from entering a forward swap contract is then given by :

$$\texttt{Payoff Forward Swap} = CB_c(T,T^*) - 100$$

where

$$CB_c(T,T^*) = \frac{c \times 100}{2} \times \sum_{j=1}^{n} B(T,T_j) + 100 \times B(T,T^*),$$

and $T_1, T_2, \ldots, T_n$ are the swap's reset dates, with $T_n = T^*$.

□ It can be seen as the payoff from entering into a Forward Contract to purchase a fixed rate bond with coupon rate c for a delivery price $K = 100$.

☐ What is the value today of this payoff ? It is the value of a forward contract to receive a coupon bond at T for a delivery price $K = 100$:

$$V^f_{swap}(0, T, T^*; c) = B(0, T) \times [\Phi^{CB}(0, T, T^*) - 100]$$

where $\Phi^{CB}(0, T, T^*) = \dfrac{c \times 100}{2} \times \sum_{j=1}^{n} \Phi(0, T, T_j) + 100 \times \Phi(0, T, T^*)$.

☐ While in a standard forward contract the delivery price K is chosen to make the value of the forward contract equal to zero at initiation, in a forward swap contract it is the swap rate c that is chosen to make the value of the contract equal to zero.

- Thus, we must look for c s.t. $V^f_{swap}(0, T, T^*; c) = 0$. The solution is:

$$f^s_2(0, T, T^*) = 2 \times \frac{1 - \Phi(0, T, T^*)}{\sum_{j=1}^{n} \Phi(0, T, T_j)}$$

  and we see that $f^s_2(0, T, T^*)$ is the swap rate implicit in the forward curve.

- **Definition:** *The* **forward swap rate** *of a forward swap contract to enter into a swap at time T, with maturity T^*, payment frequency ℓ, and payment dates T_1, T_2, ..., $T_n = T^*$ is given by*:

$$f^s_\ell(0, T, T^*) = \ell \times \frac{1 - \Phi(0, T, T^*)}{\sum_{j=1}^{n} \Phi(0, T, T_j)}$$

6.7 Futures Contracts and Futures Prices

☐ A **Futures contract** is an agreement, signed at the initial date $t = 0$, to buy or sell a **standardized** asset or a specific interest rate, at a given date T in the future, for a given price called the **Futures price**.

☐ At any date t, the futures price, **denoted** $\mathbb{F}(t, T)$, is the delivery price applicable to the futures contract. It is similar to Forward contract, yet there are some differences:

- It is traded on a *regulated exchange*, such as **Chicago Board of Trade (CBOT)** or the **Chicago Mercantile Exchange (CME)**.

- The *regulated exchange* defines the characteristics of the contract, and it guarantees that the payments will be honored at maturity, through the exchange clearinghouse.
- The security underlying the contract is **standardized**, in the sense that the future contract *clearly specifies* the type of security this is eligible for delivery, as well as the time and the method of delivery of the security.
- **Positions in Futures contracts** are governed by the so called **"marking to market"** procedure : the futures contract is worth zero at $t = 0$, then each investor is required at **every trading day** t to deposit funds into a margin account.

- Marking to market means that Profit and Losses from a futures contract' s trading activity accrues to traders with daily frequency:
  - In order to enter into a futures position, a trader must post an initial amount of money in a specific account in the exchange, called **initial margin**.
  - As the futures price moves, the margin account gets debited or credited, depending on the movement.
  - If the total amount in the account moves below the **maintenance margin**, the exchange issues a **margin call** and the trader must replenish the trading account back to the initial margin.

- Considering the futures price process $\mathbb{F} = \{\mathbb{F}(0,T), \ldots, \mathbb{F}(T,T)\}$, at the end of each trading day t (with $1 \leq t \leq T$) the balance of the investor's margin account is adjusted by the amount $\Delta_t = \mathbb{F}(t,T) - \mathbb{F}(t-1,T)$.

- Consequently, futures contracts are actually closed out after each trading day, and then start afresh the next trading day.

- The futures price at date t is given by :

$$\mathbb{F}(t,T) \;=\; E_t\left[\frac{M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}}{\exp(-r_t-\ldots-r_{T-1})}\,V_T\right] = E_t^{\mathbb{Q}}\left[V_T\right], \tag{11}$$

- and satisfies the recursive relation :

$$\mathbb{F}(t,T) \;=\; E_t\left[\frac{M_{t,t+1}}{B(t,t+1)}\,\mathbb{F}(t+1,T)\right] = E_t^{\mathbb{Q}}\left[\mathbb{F}(t+1,T)\right]. \tag{12}$$

□ So the Futures Price process $\{\mathbb{F}(t,T)\}_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale or, in other words, the process $(\Delta_t)_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale difference.

$a)$ **Proof :** At the trading day $T-1$, after the resettlement payment, the price of the futures contract is equal to zero; this means that given a pricing kernel $M_{T-1,T}$ for the period $(T-1,T)$ and, under the assumption of absence of arbitrage opportunities, we have :

$$\begin{aligned} 0 &= E_{T-1}[M_{T-1,T}(V_T - \mathbb{F}(T-1,T))] = E^{\mathbb{Q}}_{T-1}[E_{T-1}(M_{T-1,T})(V_T - \mathbb{F}(T-1,T))] \\ &= E_{T-1}\left(M_{T-1,T}\right) E^{\mathbb{Q}}_{T-1}\left(V_T - \mathbb{F}(T-1,T)\right) ; \end{aligned} \tag{13}$$

and therefore :

$$\mathbb{F}(T-1,T) = E^{\mathbb{Q}}_{T-1}\left(V_T\right) . \tag{14}$$

b) Similarly, at any date t, the price of the futures contract after the settlement is zero and, since the payoff is $\mathbb{F}(t+1,T) - \mathbb{F}(t,T)$ at date $t+1$, we have:

$$\mathbb{F}(t,T) = E_t^{\mathbb{Q}}\left(\mathbb{F}(t+1,T)\right) = E_t^{\mathbb{Q}}\left(V_T\right) . \tag{15}$$

c) Under the historical probability $\mathbb{P}$, relation (15) can be written as :

$$\begin{aligned} \mathbb{F}(t,T) &= E_t\left[\frac{M_{t,t+1}\cdot \ldots \cdot M_{T-1,T}}{E_t(M_{t,t+1})\cdot \ldots \cdot E_{T-1}(M_{T-1,T})}\ V_T\right] \\ &= E_t\left[\exp(\textstyle\sum_{i=t}^{T-1} r_i)\, M_{t,t+1}\cdot \ldots \cdot M_{T-1,T}\ V_T\right] \end{aligned}$$

□ Finally, let us give a general relation between the forward and futures price:

$$\Phi(t,T) - \mathbb{F}(t,T) = \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\Pi_{s=t}^{T-1} \exp(-r_s)\,,\, V_T\right)}{B(t,T)} \tag{16}$$

□ **Proof** - from the definition of Forward and Future price we have:

$$\begin{aligned}
\Phi(t,T) - \mathbb{F}(t,T) &= \frac{E_t^{\mathbb{Q}}\left[\exp(-\sum_{i=t}^{T-1} r_i)\, V_T\right]}{B(t,T)} - E_t^{\mathbb{Q}}(V_T) \\
&= \frac{E_t^{\mathbb{Q}}\left[\exp(-\sum_{i=t}^{T-1} r_i)\, V_T\right] \; - \; E_t^{\mathbb{Q}}\left[\exp(-\sum_{i=t}^{T-1} r_i)\right]\; E_t^{\mathbb{Q}}(V_T)}{B(t,T)} \\
&= \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left[\exp(-\sum_{i=t}^{T-1} r_i)\,,\; V_T\right]}{B(t,T)}
\end{aligned}$$

□ Therefore the forward price and the futures price coincide if and only if the two random variables $\exp(-\sum_{i=t}^{T-1} r_i)$ and V_T are $\mathbb{Q}$-uncorrelated. In particular, this is true in the case of deterministic short rates r_t.

6.7 Futures on Bonds

☐ The time t Futures Price for a Futures Contract on a zero-coupon bond maturing at date $S \geq T$, with delivery date T, is given by:

$$\mathbb{F}(t,T,S) = E_t^{\mathbb{Q}}[\, B(T,S)] \; . \tag{17}$$

☐ The Future Price at date t of a Futures Contract (with delivery date T) written on a coupon bond with payments C_i at time T_i, $i \in \{1,\ldots,n\}$ $(t < T < T_1 < \ldots < T_n = \widetilde{T})$ is given by:

$$\mathbb{F}^{CB}(t,T) \;=\; E_t^{\mathbb{Q}}\left[\, CB(T,\widetilde{T})\right] = E_t^{\mathbb{Q}}\left[\sum_{i=1}^{n} C_i\, B(T,T_i)\right] = \sum_{i=1}^{n} C_i\, \mathbb{F}(t,T,T_i)\,. \tag{18}$$

where $\mathbb{F}(t,T,T_i)$ is the Futures price on a ZCB maturing at date T_i.

□ Example : Pricing a Futures on Bonds with Gaussian AR(1) ATSM

- Let us denote the risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, in the following way:

$$
\begin{aligned}
E_t^{\mathbb{Q}}[\exp(u x_{t+1})] &= \exp\left[u(\nu^* + \varphi^* x_t) + \tfrac{1}{2}u^2\sigma^2\right] \\
&= \exp\left[a^*(u)\, x_t + b^*(u)\right] ,
\end{aligned}
$$

where $a^*(u) = u\,\varphi^*$ and $b^*(u) = u\,\nu^* + \frac{1}{2}u^2\sigma^2$.

- Then, the time t futures price for a futures on a ZCB maturing at date $S \geq T$ is given by (denoting a^{*oj} the function a^* compounded j times):

$$
\begin{aligned}
\mathbb{F}_t = E_t^{\mathbb{Q}}[B(T,S)] &= E_t^{\mathbb{Q}}\left[\exp\left(c_{S-T}\, x_T + d_{S-T}\right)\right] \\
&= E_t^{\mathbb{Q}}\left(E_{T-1}^{\mathbb{Q}}\left[\exp\left(c_{S-T}\, x_T + d_{S-T}\right)\right]\right. \\
&= \exp\left(a^{*o(T-t)}(c_{S-T})\, x_t + d_{S-T} + \textstyle\sum_{j=0}^{T-t-1} b^*\left(a^{*oj}(c_{S-T})\right)\right) .
\end{aligned}
$$

Fixed Income and Credit Risk

Lecture 6 - Part II

An Introduction to Credit Risk

Modelling and Pricing

Outline of Lecture 6 - Part II

6.9 Credit Risk

6.9.1 A Definition of Credit Risk

6.9.2 Credit Derivatives

6.9.3 Components of the expected loss in a simple static setting

6.9.4 Defaultable Bond Prices in a simple risk-free dynamic setting

6.9.5 Credit Risk : Sources of Information

6.10 Rating Based Models

6.11 Reduced Form Credit Models in Discrete-Time

6.11.1 Motivations

6.11.2 The Setup

6.11.3 The Term Structure of Corporate Bonds

6.9 Credit Risk

6.9.1 A Definition of Credit Risk

- **What is credit risk ?**

  It is the risk of not receiving an amount of money you were promised.

- We can equivalently talk about **default risk**, i.e. the risk that an obligor does not honour his payment obligations.

□ Typically:

- Default events are rare (on average over a large sample).
- They may occur unexpectedly (...besides when we are in crises periods).
- Default events involve significant losses.
- The size of these losses is unknown before default.

□ All **payment obligations** represent some sort of default risk.

☐ **Recovery :**

- We have bought a bond from a firm making default before we receive all cash flows. How much do we receive in that case? The firm is not in the possibility to pay all creditors. The **seniority** order of reimbursement typically is : i) *Senior Secured*, ii) *Senior Unsecured*, iii) *Subordinated*, iv) *Junior Unsecured*.
- The **Recovery Rate** is the fraction of the owed amount of money recovered by the creditor (i.e. the lender). It is usually expressed as a % of the owed notional, and it is not known in advance.
- **Loss Given Default =** 1 - Recovery Rate.

☐ **Components of Credit Risk**

- **Arrival risk :** whether default will occur or not $\Rightarrow$ we talk about **Probability of Default**.

  $\rightarrow$ We do not know in advance if a firm will make default or not over a time period of interest $[0, T]$. The *default event* is a stochastic phenomenon.

- **Timing risk :** the moment/date when default occurs.

  $\rightarrow$ Even if we knew that a firm will make default, we do not know when (at which date $t \in (0, T]$) this default will arrive.

  $\rightarrow$ The *time of default* is stochastic.

☐ **Components of Credit Risk**

- **Recovery risk :** how severe the losses are $\Rightarrow$ clearly, it depends on the probability distribution of the recovery rate.
- **Market risk :** changes in the market price of a defaultable asset. Several factors may affect in a "bad way" the price of the defaultable asset we have bought: i) *Common Factors* in the economy where the firm is; ii) *Specific Factors* of the industry in which the firm operates.
- **Default correlation risk :** risk of several obligors defaulting jointly (at the same time) $\Rightarrow$ joint arrival risk, joint timing risk.

6.9.2 Credit Derivatives

☐ A first definition :

- it is derivative security that is primarily used to transfer, hedge or manage credit risk.
- A derivative security whose payoff is substantially affected by credit risk.
- Examples : a) ZCB or a coupon bonds issued by a firm with a positive probability of default; b) a derivative written on such a bond.

☐ Narrower definition :

- A **credit derivative** is a derivative security that has a payoff which is conditioned on the occurrence of a **credit event**.
- The credit event is defined with respect to a **reference credit**, and the **reference credit asset(s)** issued by the reference credit.
- If the credit event has occurred, the default payment is (supposed to be) made by one of the counterparties.
- Besides the default payment, a credit derivative can have further payoffs that are not default contingent.

☐ **Market terminology :**

- Buying a credit derivative typically means **buying credit protection**, which is economically equivalent to **shorting the credit risk**.
- So **selling** credit protection means going **long** the credit risk.
- Alternatively, one may speak of **protection buyers/sellers** as the **payers/receivers** of the premium.

☐ **Credit Event :** an event that gives a protection buyer the right to settle a credit derivative. Examples :

- **bankruptcy** $\rightarrow$ defined quite widely to include the credit reference being dissolved, becoming insolvent, making an arrangement for the benefit of its creditors, and having a judgment of insolvency made against it.

- **failure to pay** $\rightarrow$ when the credit reference fails to make interest (coupons for instance) or principal (face value) payments when due after a permitted *grace period*.

  $\hookrightarrow$ a grace period is a period, usually of 30 days, in which a borrower is permitted to make interest or principal payments that it has missed.

☐ Examples (continued) :

- **restructuring** $\rightarrow$ when the credit reference restructures its debt. For instance, we have interest payments reduced, principal amount reduced, maturity extended, becoming subordinated to another obligation or having its currency changed.
- **ratings downgrade** (by rating Agency) below given threshold $\rightarrow$ the downgraded firm is judged riskier and this information is discounted by the market (which ask for a compensation) by selling stocks. Thus, the quoted stock price $\downarrow$.

- Let us consider again a firm that has a given probability of defaulting and not paying all it borrowed. The firm can borrow money from a bank (**loans**) or from investors (**bonds**).
- So What? **Where does credit risk show up?**
- Credit risk will determine the borrowing costs of that company:
  - The firm will have to pay higher coupon on its bonds in order for investors to be willing to buy them. That is, in order to compensate for the risk.
  - Banks will require higher interests on the loan to be compensated for the higher risk.

6.9.3 Components of the expected loss in a simple static setting

- ☐ Any institution (firm/bank) is interested in evaluating its **Expected capital Loss** over its credit portfolio during a time interval of interest $[0, T]$.
- ☐ The capital associated to the Expected Loss is supposed to be compensated (hedged) by a given amount of **Reserves**.
- ☐ For any given credit, the associated Expected Loss is function of: i) the **default probability**; ii) the exposure at the date of default (**Exposure-at-Default**), i.e. the residual amount of money to be paid at the default date, and the loss depending on the recovery rate.

□ Now, the realized loss may differ from the expected loss because of uncertainty. A bank, for instance, is worried not only about the unexpected but also about the expected loss.

□ The bank is interested to know (to anticipate) the maximum amount of expected (potential) loss over a given time horizon of interest.

□ Let us focus on the Expected Loss, and let us study its components. We consider a loan of an amount A_0 that a firm obtain at date $t = 0$ from the bank, at an annual rate $R_{0,T}$, over the period $[0, T]$.

□ In the case of **no default**, the value at date T of this credit is:

$$A_T = A_0 \times (1 + R_{0,T})^T \, .$$

□ In the case of **default**, the value at date T of this credit is a fraction RR_T of $A_0 \times (1 + R_{0,T})^T$:

$$A_T = A_0 \times (1 + R_{0,T})^T RR_T \, ,$$

where RR_T is the recovery rate at T.

□ At date $t = 0$, there is uncertainty about the default event in $t = T$ and about the size of RR_T in case of default. Let us define:

$$Z_T = \begin{cases} 1 & \text{if default in } t = T \\ 0 & \text{if not} \, . \end{cases}$$

☐ The value of the credit at $t = T$ is therefore given by:

$$\begin{aligned} A_T &= A_0 \times (1 + R_{0,T})^T - Z_T[A_0 \times (1 + R_{0,T})^T(1 - RR_T)] \\ &= A_0 \times (1 + R_{0,T})^T[1 - Z_T(1 - RR_T)]\,, \end{aligned}$$

☐ In the terminology of Basel Committee, we have:

$$\begin{aligned} EAD_T &= A_0 \times (1 + R_{0,T})^T \text{ is the Exposure-at-Default} \\ RR_T &= \text{ is the Recovery Rate} \\ 1 - RR_T &= \text{ is the Loss-Given-Default}\,. \end{aligned}$$

☐ The expected value of the credit is therefore given by:

$$\begin{aligned} E_0(A_T) &= A_0 \times (1 + R_{0,T})^T - E_0[Z_T[A_0 \times (1 + R_{0,T})^T(1 - RR_T)]] \\ &= A_0 \times (1 + R_{0,T})^T[1 - E_0[Z_T(1 - RR_T)]]\,, \end{aligned}$$

☐ while, the expected loss is:

$$\begin{aligned} E_0(L_T) &= A_0 \times (1 + R_{0,T})^T \times E_0[Z_T(1 - RR_T)] \\ &= EAD_T \times E_0[LGD_T \,|\, Z_T = 1] \times E_0(Z_T) \\ &= EAD_T \times ELGD_T \times DP_T\,, \end{aligned}$$

☐ given that:

$$\begin{aligned} E_0[Z_T(1 - RR_T)] &= E_0[Z_T(1 - RR_T) \,|\, Z_T = 1] \times P_0(Z_T = 1) \\ &\quad + E_0[Z_T(1 - RR_T) \,|\, Z_T = 0] \times P_0(Z_T = 0) \\ &= E_0[(1 - RR_T) \,|\, Z_T = 1] \times P_0(Z_T = 1)\,. \end{aligned}$$

☐ We have that $DP_T = E_0(Z_T) = P_0(Z_T = 1)$ is the **Default Probability** and $ELGD_T = E_0[LGD_T \,|\, Z_T = 1]$ is the **Expected Loss-Given-Default** .

☐ **Remark :** Observe that in this simple case, we have not corrected for the discount rate between 0 and T ($M_{0,T} = 1$).

6.9.4 Defaultable Bond Prices in a simple risk-free dynamic setting

- ☐ How do we translate the credit risk into a bond price specification ?
- ☐ Let us consider the *old* problem to price $DB(0,T)$ at date $t=0$ a ZCB maturing at date $T>0$ (with unitary face value). But now this asset is issued by a firm characterized by a positive probability of default.
- ☐ It is clear that $DB(t,T)$ is going to also depend on i) probability that the firm make default before T and on ii) the recovery rate in case of default.

□ **Notation and assumptions - 1 :**

- τ denotes the random time of default; if $\tau > T \Rightarrow$ no default.
- τ is a non-negative random variable characterized by an Exponential Law with parameter λ:

  i) $f_\lambda(\tau)\, d\tau = \lambda \text{ exp}(-\lambda\, \tau)\, d\tau\, \mathbb{I}_{(\tau \geq 0)}$ is the $\mathbb{P}\{$Default occurs at some $\tau\}$;

  ii) $\mathbb{P}(\tau > T \,|\, \tau > t) = \text{exp}[-\lambda\,(T - t)] = \mathbb{P}\{$Survival over $[t, T]\}$

  iii) $\mathbb{P}(\tau \leq t) = 1 - \text{exp}(-\lambda\, t) = 1 - \mathbb{P}\{$Survival over $[0, t]\}$

  iv) $\lim_{s \to 0} s^{-1}\mathbb{P}(\tau > t + s \,|\, \tau > t) = \lambda \;\; \forall \;\; t > 0$ is the default intensity.

□ Notation and assumptions - 2 :

- given the default time τ, we define (again) the default process by:

$$Z_{\tau,T} = \begin{cases} 1 \text{ if default at } \tau \leq T \\ 0 \text{ if not} . \end{cases}$$

  It is a point process with one jump of size one at default.

- RR denotes the recovery rate and it is constant.

- the default event can happen at dates $t_1 < t_2 < \ldots < t_N = T$.

- we assume a risk-free discount factor ($\mathbb{P} = \mathbb{Q}$) with a constant short rate ($r_t = r$) : $M_{0,t} = DF_{0,t} = \exp(-r\,t)$.

□ Default and Survival Probabilities :

- Let us assume to be at date $t \in (0, T)$ and the firm is alive.
- We assume that the probability that the firm make default by time T, given that we know that at t is alive is given by:

$$\mathbb{P}(\tau \leq T \,|\, \tau > t) = \mathbb{P}(\tau \leq T - t) = 1 - e^{-\lambda\,(T-t)} = DP_{\tau,t,T}$$

- The **survival probability** $\mathcal{S}_{\tau,t,T}$ is given by:

$$\mathcal{S}_{\tau,t,T} = \mathbb{P}(\tau > T \,|\, \tau > t) = e^{-\lambda\,(T-t)} = 1 - \mathbb{P}(\tau \leq T - t)$$

□ **Pricing-1 :**

- The unitary payoff, of the defaultable ZCB with date t price $DB(t,T)$, is:

$$\begin{aligned} y(T) &= RR\,\mathbb{I}_{\{\tau\leq T\}} + \mathbb{I}_{\{\tau>T\}} \\ &= RR\,Z_{\tau,T} + (1 - Z_{\tau,T}) \\ &= 1 - Z_{\tau,T}\,(1-RR)\,. \end{aligned}$$

- Following the A.A.O. principle, under the risk-neutral probability measure $\mathbb{Q}=\mathbb{P}$ we can write:

$$\begin{aligned} DB(0,T) &= \exp(-\,r\,T)E_0^{\mathbb{P}}[1 - Z_{\tau,T}\,(1-RR)] \\ &= \exp(-\,r\,T) - \exp(-\,r\,T)\,(1-RR)\,\mathbb{P}(\tau\leq T-t) \\ &= \exp(-\,r\,T) - \exp(-\,r\,T)\,(1-RR)\,DP_{\tau,t,T}\,. \end{aligned}$$

where $B(0,T) = \exp(-\,r\,T)$ is the non-defaultable ZCB price and the second term is the **risk-free discounted value** of the **expected loss**.

□ Pricing-2 :

- Indeed, we have that $EAD_T = 1$, $LGD_T = (1 - RR)$ and $E_0(Z_{\tau,T}) = DP_{\tau,t,T}$. Thus, the loss at date T is $L_T = (1 - RR)\, Z_{\tau,T}$ and therefore $E_0(L_T) = (1 - RR)\, DP_{\tau,T}$.

- We have that the difference between the risk-free (non defaultable) and the risky (defaultable) bond price is (in that simple setting) given by

$$\exp(-\, r\, T)\,(1 - RR)\, DP_{\tau,t,T},$$

that is the present value of the expected loss (associated to one unit of money).

☐ Observe that :

- If $RR = 0$, then $y(T) = 1 - Z_{\tau,T}$ and $DB(0,T) = \exp(-\,r\,T)\,(1 - DP_{\tau,t,T})$.
- if $DP_{\tau,t,T} = 0$ (or $RR = 1$) then $DB(0,T) = B(0,T)$.
- The ZCB price $DB(0,T)$ **increases** as $DP_{\tau,T}$ **decreases**
- The ZCB price $DB(0,T)$ **increases** as RR **increases** toward one.

☐ Remember the assumptions :

- $\mathbb{Q} = \mathbb{P}$; • recovery and short rates are assumed constant;
- default probability, recovery rate and discount factors are independent;

- Relaxing these assumptions drives toward more realistic and more complicate pricing formulas.
- In general (but assuming a recovery $RR_{t,t+h}$ at maturity even if the default date is at $\tau < t+h$) we can always write:

$$
\begin{aligned}
DB(t,t+h) &= E_t^{\mathbb{P}}[M_{t,t+h}\,(RR_{t,t+h}\mathbb{I}_{\{\tau\leq t+h\}} + \mathbb{I}_{\{\tau>t+h\}})] \\
&= E_t^{\mathbb{P}}[M_{t,t+h} \times RR_{t,t+h} \times \mathbb{I}_{\{\tau\leq t+h\}}] + E_t^{\mathbb{P}}[M_{t,t+h} \times \mathbb{I}_{\{\tau>t+h\}}] \\
&= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{t+h-1} r_i\right) \times RR_{t,t+h} \times \mathbb{I}_{\{\tau\leq t+h\}}\right] \\
&\qquad + E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{t+h-1} r_i\right) \times \mathbb{I}_{\{\tau>t+h\}}\right]
\end{aligned}
$$

□ If, in addition, we assume $RR_{t,t+h} = 0$:

$$
\begin{aligned}
DB(t,t+h) &= E_t^{\mathbb{P}}[M_{t,t+h} \times \mathbb{I}_{\{\tau>t+h\}}] \\
&= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{t+h-1} r_i\right) \times \mathbb{I}_{\{\tau>t+h\}}\right]
\end{aligned}
$$

□ Even more generally (assuming a recovery $RR_{t,t+i}$ at the default date $\tau = t+i$)

we can always write:

$$
\begin{aligned}
DB(t,t+h) &= \sum_{i=1}^{h} E_t^{\mathbb{P}}[M_{t,t+i} \times RR_{t,t+i} \times \mathbb{I}_{\{\tau=t+i\}}] + E_t^{\mathbb{P}}[M_{t,t+h} \times \mathbb{I}_{\{\tau>t+h\}}] \\
&= \sum_{i=1}^{h} E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=t}^{t+i-1} r_j\right) \times RR_{t,t+i} \times \mathbb{I}_{\{\tau=t+i\}}\right] \\
&\quad + E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{t+h-1} r_i\right) \times \mathbb{I}_{\{\tau>t+h\}}\right]
\end{aligned}
$$

□ Nevertheless, if we assume $RR_{t+i} = 0$:

$$\begin{aligned} DB(t,t+h) &= E_t^{\mathbb{P}}[M_{t,t+h} \times \mathbb{I}_{\{\tau>t+h\}}] \\ &= E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=t}^{t+h-1} r_i\right) \times \mathbb{I}_{\{\tau>t+h\}}\right] \end{aligned}$$

☐ We have to make assumptions about:

- the information used by the investor to price: which variables enter into the factor x_t ?

$$\begin{aligned} x_t &= \text{observables and/or latent factors} \\ &= \text{financial and/or macro factors} \\ &= \text{common and/or specific factors} \end{aligned}$$

- which are the assumptions about $M_{t,t+h}$ and the associated market price of factor risks ?

- which is the joint historical dynamics of the SDF, default process and recovery rate ?

6.9.5 Credit Risk : Sources of Information

□ **Credit risk = default probabilities over a given horizon** AND **recovery rate conditional on default**

□ **Challenges :**

- *Asymmetry* - limited upside but substantial downside,
- Credit returns are *skewed* and *fat-tailed*,
- *Defaults are rare events* - difficult to estimate default probabilities or default correlations from data.

☐ Where can we find information about the credit risk components ?

☐ 3 main sources of information :

- **from RATINGS**,
- **from the FIRM**,
- **from the MARKET**.

☐ **Go to Rating Agencies** and search for credit ratings (grades!).

☐ **Go to the firm** and search for the true determinants of default and recovery rate dynamics:

- value of assets;
- balance sheets;
- amount of debt, interest rates, ...;
- Expected cash flows, ...;

☐ **Go to the market** (reverse engineering)

- Prices should tell us something about the firms credit risk (market assessment of credit risk).

☐ Each source of information **identify** a modelling approach:

- **Rating Based Models : from the Rating Agency**
- **Structural Models : from the FIRM**. Credit risk comes from the firms fundamentals.
- **Reduced Form Models : from the MARKET**. Credit risk comes from the market assessment.

□ Rating Based Models :

- Rely on credit ratings provided by rating agencies:

Fitch, S & P	AAA	AA	A	BBB	BB	B	CCC	CC	C	D	NR
Moody's	Aaa	Aa	A	Baa	Ba	B	Caa	Ca	C	D	NR

  where AAA denotes the best rating (grade), D indicate Default and NR means Not Rated.

- From AAA (Aaa) to BBB (Baa) we have an **Investment Grade** (high quality). From BB (Ba) to C we have a **Speculative Grade** ("Junk").

- Assumption: all relevant information (for credit risk) is captured by ratings categories and the probabilities of transiting from one category to another.

☐ **Structural Models :**

- They directly relate default probabilities and recovery rates to firm fundamentals.
- Favored by the part of the academia (finance/economics) interested to explain the structural (form-based) reasons of the credit risk.
- Hard to calibrate (properly).
- Consistent framework to price equity-credit products.
- In the industry several simple structural models are used (though most people are not aware of)

□ **Reduced Form Models :**

- Default modeled using an exogenous default intensity process.
- Popular within the industry for their easy and quick calibration to market prices.
- They have the (quasi) monopoly in defaultable bond pricing and credit derivative pricing (like Credit Default Swap).
- Favored by the part of the academia (finance/economics) interested to well fit the data (model flexibility).

6.10 Rating Based Models

☐ Ratings agencies, such as Standard & Poor's or Moody's KMV issue ratings on the creditworthiness of borrowers:

- they use historical data on defaults over a period of more than 20 years.
- Debtors are considered in default as soon as they miss a payment obligation on any coupon or principal.
- Based on historical data, ratings agencies estimate probability p_{ij} of transiting from ratings category i to ratings category j over a give horizon.

2008 Global Corporate Transition Rates (%)

From/To	AAA	AA	A	BBB	BB	B	CCC/C	D	NR
AAA	81.82	6.06	3.03	0.00	0.00	1.01	2.02	0.00	6.06
AA	0.00	77.65	17.23	0.57	0.00	0.00	0.19	0.38	3.98
A	0.00	1.59	87.59	4.92	0.45	0.00	0.00	0.38	5.07
BBB	0.00	0.00	2.57	86.81	3.59	0.27	0.20	0.47	6.09
BB	0.00	0.09	0.00	4.94	77.21	8.26	1.04	0.76	7.69
B	0.00	0.00	0.00	0.14	3.68	73.16	8.08	3.82	11.11
CCC/C	0.00	0.00	0.00	0.00	0.00	11.22	41.84	26.53	20.41

Source: Standard & Poor's Global Fixed Income Research and Standard & Poor's CreditPro®.

☐ Limitations:

- Averages across heterogeneous firms (different industries etc.)
- Averages over the business cycle, but there are significantly more defaults in recessions than there are in booms.
- Recovery rates also vary across business cycle.

☐ For developments: see Lando (2004) and the references therein, Foulcher, Gourieroux and Tiomo (2004, 2006), Gagliardini and Gourieroux (2005a, 2005b).

6.11 Reduced Form Credit Models in Discrete-Time

6.11.1 Motivations

□ The fundamental reference is the paper Gourieroux, Monfort and Polimenis (2006): "Affine Models for Credit Risk Analysis", *Journal of Financial Econometrics*.

□ Propose a pricing method for corporate bonds (among other) which would be :

- **tractable** (almost explicit pricing formulas)
- in **discrete time** and **sufficiently flexible**.

☐ and which would take into account:

- **a time and age varying default risk**;
- **a correlation** between **default** and **default free term structure**;
- **default correlation**;
- see GMP (2006, JFEC) for details.

- See also Mueller (2009) and Monfort and Renne (2010).

6.11.2 The Setup

- ☐ We have n firms $i \in \{1, \ldots, n\}$ and we denote by τ_i the failure date for any firm $i \in \{1, \ldots, n\}$.

- ☐ **Assumption A.1:** There exist general (systematic) and corporate specific **factors**, respectively denoted by $(x_t), (x_t^i) = i = 1, \ldots, n$. These factors are independent, Markovian and their transitions are such that :

$$E[\exp(u' x_{t+1}) \,|\, \underline{x_t}] = \exp[a_g(u)' x_t + b_g(u)]$$

$$E[\exp(u' x_{t+1}^i) \,|\, \underline{x_t^i}] = \exp[a_c(u)' x_t^i + b_c(u)]$$

☐ Thus, the factors satisfy a compound autoregressive (Car) process [see Darolles, Gourieroux, Jasiak (2006)]. The conditional distributions are defined by means of the conditional Laplace transform, or moment generating function, restricted to real arguments u.

☐ By Assumption A.1. the population is assumed homogenous, that is, the distributions of the corporate specific factor processes are independent of the firm.

☐ Thus, the cohort is both homogenous with respect to the birth date and to individual characteristics such as the industrial sector, or the initial rating.

- Finally, the general factor x_t is defined for any date t, whereas the firm specific factor x_t^i only exists until the default date τ_i of the i^{th} firm.
- This explains why the independence between idiosyncratic and systematic factors is assumed.
- Otherwise, complicated effects have to be taken into account at any firm's failure time [see e.g. Jarrow, Yu (2001), Gagliardini, Gourieroux (2003) for this extension in the case of two borrowers].

☐ In this model, instantaneous default correlation arises only because of the common risk factors that drive individual firms' default intensities. Equivalently, given those common factors, default arrivals of different firms become independent.

☐ Notation : $\underline{x} = (x_t, \forall t)$, $\underline{x^i} = (x_t^i, \forall t)$, and, $\underline{x_h} = (x_t, t \leq h)$, $\underline{x_h^i} = (x_t^i, t \leq h)$.

☐ **Assumption A.2:** Conditional on the realization path of the factors, $\underline{x}, \underline{x^i}, i = 1, \ldots, n$, default arrival times $\tau_i, i = 1, \ldots, n$ are independent.

☐ Moreover, the conditional survivor intensities are such that :

$$\begin{aligned}\mathcal{S}^i_{\tau,t,t+1} &= \mathbb{P}[\tau_i > t+1 | \tau_i > t, \underline{x}, \underline{x^j}, j = 1,\dots,n] \\ &= \mathbb{P}[\tau_i > t+1 | \tau_i > t, x_{t+1}, x^i_{t+1}] \\ &= \exp[-(\zeta_{t+1} + \beta'_{t+1} x_{t+1} + \gamma'_{t+1} x^i_{t+1})] \\ &= \exp(-\lambda^i_{t+1}), \text{ say}, \forall t,\end{aligned}$$

☐ where $\zeta_{t+1}, \beta_{t+1}, \gamma_{t+1}$ are deterministic functions of time.

☐ Since the conditional survivor probability is smaller than 1, we have : $\lambda^i_t = \zeta_t + \beta'_t x_t + \gamma'_t x^i_t \geq 0, \forall t$. These restrictions imply conditions on both the sensitivity parameters and the factor distribution. For instance, they are satisfied if both factors and sensitivity coefficients are nonnegative.

- The survivor intensity depends on time by means of factors x_{t+1}, x^i_{t+1} and sensitivities $\zeta_{t+1}, \beta_{t+1}, \gamma_{t+1}$.
- The sensitivities (factor loadings) $\zeta_{t+1}, \beta_{t+1}, \gamma_{t+1}$ capture the age effect : their dependence on $(t+1)$ catch information about the age of the living firm.
- The general factor x_t, and the specific factor x^i_t capture the time effect : they account for an economy or an industry in recession or expansion.

☐ **Assumption A.3:** We are under the absence of arbitrage opportunity principle, and we assume the one-period stochastic discount factor $M_{t,t+1}$ is given by:

$$M_{t,t+1} = \exp\left[\alpha_t(\underline{x}_t)' x_{t+1} + \delta_t(\underline{x}_t)\right] ,$$

where α_t is the "factor loading" or "sensitivity" vector.

☐ Since $\exp(-r_t) = E_t(M_{t,t+1}) = \exp\left[\psi_t(\alpha_t \,|\, \underline{x}_t) + \delta_t\right]$, the SDF can also be written:

$$\begin{aligned} M_{t,t+1} &= \exp\left[-r_t + \alpha_t'(\underline{x}_t) x_{t+1} - \psi_t(\alpha_t | \underline{x}_t)\right] \\ &= \exp\left[-(\theta_0 + \theta_1{}' x_t) + \alpha_t'(\underline{x}_t) x_{t+1} - \psi_t(\alpha_t | \underline{x}_t)\right] , \end{aligned}$$

where $\psi_t(u) = \log \varphi_t(u) = \log E_t[\exp(u' x_{t+1})]$ denotes the historical conditional log-Laplace transform of the factor (x_t), and where we have assume (as usual) $r_t = \theta_0 + \theta_1{}' x_t$.

☐ When the SDF is exponential-affine, we have convenient additional results:

$$\frac{d\,\mathbb{Q}_{t,t+1}}{d\,\mathbb{P}_{t,t+1}} = d_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t}) = \frac{\exp(\alpha_t' x_{t+1} + \delta_t)}{E_t \exp(\alpha_t' x_{t+1} + \delta_t)}$$

$$= \exp\left[\alpha_t' x_{t+1} - \psi_t(\alpha_t)\right],$$

so $d_t^{\mathbb{Q}}$ is also exponential-affine.

☐ The conditional R.N. Laplace transform of the factor x_{t+1}, given $\underline{x_t}$, is :

$$\varphi_t^{\mathbb{Q}}(u|\underline{w_t}) = E_t^{\mathbb{Q}}\left[\exp(u' x_{t+1})\right] = E_t \exp\left[(u + \alpha_t)' x_{t+1} - \psi_t(\alpha_t)\right]$$

$$= \frac{\varphi_t(u + \alpha_t)}{\varphi_t(\alpha_t)}$$

☐ and, consequently, the associated conditional R.N. Log-Laplace transform is :

$$\psi_t^{\mathbb{Q}}(u) = \psi_t(u + \alpha_t) - \psi_t(\alpha_t)\,.$$

□ Conversely, we get :

$$\frac{d\,\mathbb{P}_{t,t+1}}{d\,\mathbb{Q}_{t,t+1}} = d_t^{\mathbb{P}}(x_{t+1}|\underline{x_t}) = \text{exp}\left[-\alpha_t' x_{t+1} + \psi_t(\alpha_t)\right]$$

□ and, taking $u = -\alpha_t$ in $\psi_t^{\mathbb{Q}}(u)$, we can write :

$$\psi_t^{\mathbb{Q}}(-\alpha_t) = -\psi_t(\alpha_t)$$

□ and, replacing u by $u - \alpha_t$, we obtain :

$$\psi_t(u) = \psi_t^{\mathbb{Q}}(u - \alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t).$$

☐ We also have :

$$d_t^{\mathbb{P}}(x_{t+1}|\underline{x}_t) = \text{exp}\left[-\alpha_t' x_{t+1} - \psi_t^{\mathbb{Q}}(-\alpha_t)\right] ,$$

$$d_t^{\mathbb{Q}}(x_{t+1}|\underline{x}_t) = \text{exp}\left[\alpha_t' x_{t+1} + \psi_t^{\mathbb{Q}}(-\alpha_t)\right] .$$

6.11.3 The Term Structure of Corporate Bonds

☐ Price at t of a payoff g_{t+h} at $t+h$:

$$\begin{aligned} P_t(g,t+h) &= E_t(M_{t,t+1}\dots M_{t+h-1,t+h}\, g_{t+h}) \\ &= E_t(M_{t,t+h}\, g_{t+h}) \end{aligned}$$

☐ **Zero-coupon non-defaultable bonds :**

$$B(t,t+h) = E_t(M_{t,t+h})$$

☐ **Zero-coupon Corporate bond issued by the firm i (zero recovery rate):**

$$DB_i(t,t+h) = E_t[M_{t,t+h}\, \mathbb{I}_{\tau_i > t+h}] = E_t[M_{t,t+h}\,(1 - Z_{\tau_i,t+h})] = E_t[M_{t,t+h}\, \mathcal{S}^i_{\tau,t,t+h}]$$

☐ **Key Idea :** we want to calculate

$$DB_i(t, t+h) = E[M_{t,t+h}\, \mathbb{I}_{\tau_i > t+h} \,|\, I_t]\,.$$

where $I_t = (\underline{x}_t, \underline{x}_t^i, \tau_i > t)$.

☐ We have that $\mathbb{I}_{\tau_i > t+h} = \prod_{j=1}^{h} \mathbb{I}_{\tau_i > t+j}$ and:

$$\begin{aligned} DB_i(t, t+1) &= E[M_{t,t+1}\, \mathbb{I}_{\tau_i > t+1} \,|\, I_t] \\ &= E[M_{t,t+1}\, E[\mathbb{I}_{\tau_i > t+1} \,|\, x_{t+1}, x_{t+1}^i, I_t] \,|\, I_t] \\ &= E_t[M_{t,t+1}\, \exp(-\lambda_{t+1}^i)]\,. \end{aligned}$$

☐ Thus, it is possible to prove (exercise):

$$\begin{aligned} DB_i(t, t+h) &= E[M_{t,t+1} \ldots M_{t+h-1,t+h} \mathbb{I}_{\tau_i > t+h}) \,|\, I_t] \\ &= E[M_{t,t+1} \ldots M_{t+h-1,t+h}\, \exp(-\lambda_{t+1}^i - \ldots - \lambda_{t+h}^i) \,|\, I_t]\,. \end{aligned}$$

☐ We can also write:

$$
\begin{aligned}
& DB_i(t,t+h) \\
= \; & E[M_{t,t+1}\, DB_i(t+1,t+h)\, \mathbb{I}_{\tau_i > t+1}) \,|\, I_t] \\
= \; & E[M_{t,t+1}\, DB_i(t+1,t+h)\, \mathbb{P}(\tau_i > t+1 \,|\, x_{t+1}, x^i_{t+1}, I_t) \,|\, I_t] \\
= \; & E[M_{t,t+1}\, DB_i(t+1,t+h)\, \exp(-\lambda^i_{t+1}) \,|\, I_t] \\
= \; & E[M_{t,t+1}\, DB_i(t+1,t+h)\, \exp(-\zeta_{t+1} - \beta'_{t+1} x_{t+1} - \gamma'_{t+1} x^i_{t+1}) \,|\, I_t] \\
= \; & E\{\exp[-\theta_0 - \theta'_1 x_t + \alpha'_t x_{t+1} - \psi_t(\alpha_t)]\, DB_i(t+1,t+h) \\
& \qquad \exp(-\zeta_{t+1} - \beta'_{t+1} x_{t+1} - \gamma'_{t+1} x^i_{t+1}) \,|\, I_t\}
\end{aligned}
$$

☐ Suggest the pricing formula $DB_i(t,t+h) = \exp[\widetilde{C}'_{h,t} x_t + \widetilde{C}^i_{h,t}{}' x^i_t + \widetilde{D}^i_{h,t}]$ and we solve recursively (as in the non-defaultable case).

☐ We find the recursive equations $\widetilde{C}_{h,t}, \widetilde{C}^i_{h,t}, \widetilde{D}^i_{h,t}$.

□ For $\lambda_t = 0$ for all t, we find the non-defaultable ZCB price $B_i(t, t+h)$. Indeed, we coherently have $\mathcal{S}^i_{\tau,t,t+1} = 1$ for any t.

□ The credit spread (in this case with zero recovery) is given by:

$$
\begin{aligned}
CS^i(t, t+h) &= R_i^D(t, t+h) - R_i(t, t+h) = -\frac{1}{h} \ln \frac{DB_i(t, t+h)}{B(t, t+h)}, \\
&= -\frac{1}{h} \ln E_t^{\mathbb{Q}^{(t+h)}}[\mathbb{I}_{\{\tau_i > t+h\}}],
\end{aligned}
$$

□ where $\mathbb{Q}^{(t+h)}$ denotes the $(t+h)$-forward neutral probability measure.

☐ **Example:** x_t and x_t^i follow scalar independent Gaussian AR(1) processes under the historical probability $\mathbb{P}$:

$$\begin{aligned} x_{t+1} &= \nu + \varphi\, x_t + \sigma\, \varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim N(0,1) \\ x_{t+1}^i &= \nu^i + \varphi^i\, x_t^i + \sigma^i\, \varepsilon_{t+1}^i\,, \;\; \varepsilon_{t+1}^i \sim N(0,1)\,. \end{aligned}$$

☐ We assume that the one-period Stochastic Discount Factor $M_{t,t+1}$ is given by:

$$M_{t,t+1} = \exp\left[-(\theta_0 + \theta_1\, x_t) + \alpha_t\, x_{t+1} - \psi_t(\alpha_t)\right]\,,$$

where $r_t = \theta_0 + \theta_1\, x_t$ is the risk-free rate, $\alpha_t = \alpha_0 + \alpha_1\, x_t$ is the market price of factor risk and $\psi_t(u) = \log \varphi_t(u) = \log E_t[\exp(u x_{t+1})]$ denotes the historical conditional log-Laplace transform of the general factor (x_t).

☐ The last assumption concerns the conditional survivor intensity, which is given by :

$$\begin{aligned} \mathcal{S}^i_{\tau,t,t+1} &= \mathbb{P}[\tau_i > t+1 | \tau_i > t, \underline{x}, \underline{x^i}] \\ &= \mathbb{P}[\tau_i > t+1 | \tau_i > t, x_{t+1}, x^i_{t+1}] \\ &= \exp[-(\zeta_{t+1} + \beta_{t+1} x_{t+1} + \gamma_{t+1} x^i_{t+1})] \\ &= \exp(-\lambda^i_{t+1}), \text{ say}, \forall t, \end{aligned}$$

where $\zeta_{t+1}, \beta_{t+1}, \gamma_{t+1}$ are deterministic functions of time. We will denote the date t information as $I_t = (\underline{x}_t, \underline{x}^i_t, \tau_i > t)$.

□ It is easy to verify (exercise) that $DB_i(t,t+h) = \exp[\widetilde{C}_{h,t}\, x_t + \widetilde{C}^i_{h,t}\, x^i_t + \widetilde{D}^i_{h,t}]$ with:

$$\begin{cases} \widetilde{C}_{h,t} &= -\theta_1 + (\widetilde{C}_{h-1,t+1} - \beta_{t+1})(\varphi + \alpha_1\, \sigma^2) \\ \widetilde{C}^i_{h,t} &= (\widetilde{C}^i_{h-1,t+1} - \gamma_{t+1})\, \varphi^i \\ \widetilde{D}^i_{h,t} &= -\theta_0 + \widetilde{D}^i_{h-1,t+1} - \zeta_{t+1} + (\widetilde{C}_{h-1,t+1} - \beta_{t+1})\,(\nu + \alpha_0 \sigma^2) \\ & \quad + \frac{1}{2}(\widetilde{C}_{h-1,t+1} - \beta_{t+1})^2 \sigma^2 + (\widetilde{C}^i_{h-1,t+1} - \gamma_{t+1})\, \nu^i \\ & \quad + \frac{1}{2}(\widetilde{C}^i_{h-1,t+1} - \gamma_{t+1})^2 (\sigma^i)^2\,. \end{cases}$$

□ with $c_{0,t} = 0$, $c^i_{0,t} = 0$, $d_{0,t} = 0$.

□ The Term Structure of Defaultable Interest Rates is given by:

$$R^D_i(t,t+h) = -\frac{1}{h}\, \widetilde{C}_{h,t}\, x_t - \frac{1}{h}\, \widetilde{C}^i_{h,t}\, x^i_t - \frac{1}{h}\, \widetilde{D}^i_{h,t}\,.$$

□ Given that $R(t, t+h)$ is given by:

$$R(t, t+h) = -\frac{1}{h}\, C_h\, x_t - \frac{1}{h}\, D_h$$

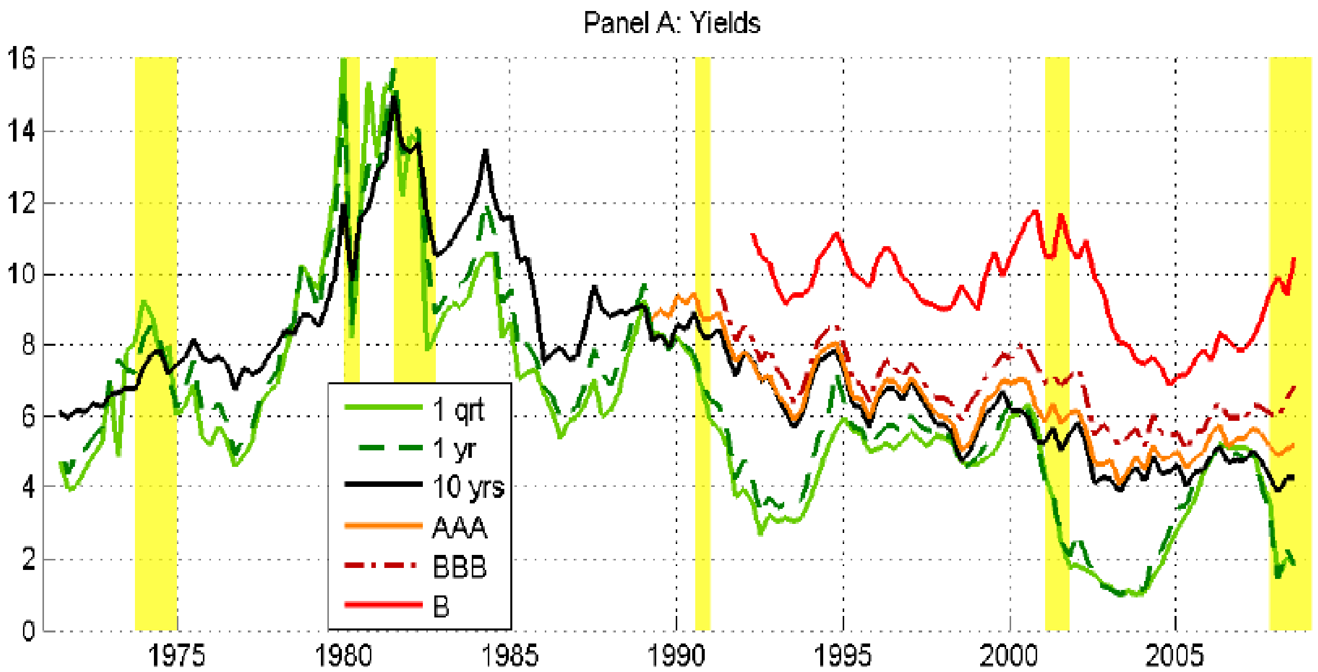
□ with recursive equations:

$$\begin{cases} C_h &= -\theta_1 + C_{h-1}(\varphi + \alpha_1\, \sigma^2) \\ D_h &= -\theta_0 + D_{h-1} + C_{h-1}\,(\nu + \alpha_0 \sigma^2) + \frac{1}{2}(C_{h-1})^2 \sigma^2\,, \end{cases}$$

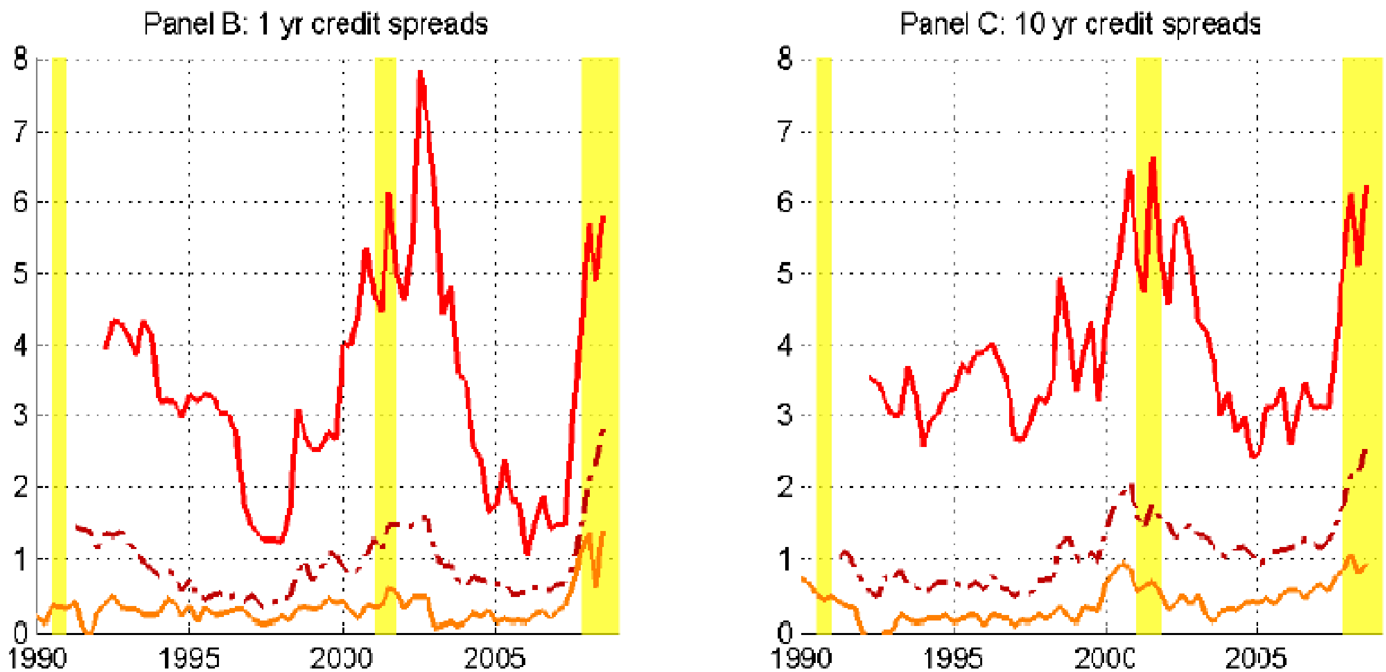
□ we have that the credit spread is still an affine function of the factors:

$$CS^i(t, t+h) = -\frac{1}{h}\,[\widetilde{C}_{h,t} - C_h]\, x_t - \frac{1}{h}\, \widetilde{C}^i_{h,t}\, x^i_t - \frac{1}{h}\,[\widetilde{D}^i_{h,t} - D_h]\,.$$

- ☐ How do defaultable (corporate) and non-defaultable yields and associated spreads move over time ? How do they behave around recessionary periods ?
- ☐ Let us see a couple of pictures taken from Mueller (2009).
- ☐ Panel A shows the 3-month, 1-year and 10-year Treasury yields, and 10-year corporate bond yields for AAA, BBB and B credits, respectively. Panels B and C show the 1-year and 10-year spreads for AAA, BBB and B credits, respectively. The time period is 1971:3 - 2008:3 although corporate bond data are only available starting in the 1990's. The shaded regions show the NBER recessions.



Panel A: Yields
16
14
12
10
8
6
4
2
0
1975
1980
1985
1990
1995
2000
2005
1 qrt
1 yr
10 yrs
AAA
BBB
B


Panel B: 1 yr credit spreads
Panel C: 10 yr credit spreads
0
1
2
3
4
5
6
7
8
1990
1995
2000
2005