Jacobian transformation

The Jacobian transformation is an algebraic method for determining the probability distribution of a variable y that is a function of just one other variable x (i.e. y is a transformation of x) when we know the probability distribution for x.

- Let x be a variable with probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$;
- Let y be another variable with probability density function $f_y(y)$ and cumulative distribution function $F_y(y)$;
- Let y be related to x by some function such that x and y increase monotonically, then we can equate changes $dF_y(y)$ and $dF_x(x)$ together, i.e.:

  $$|f_y(y)dy| = |f_x(x)dx|$$

  Rearranging a little, we get:

  $$f_y(y) = \left|\frac{dx}{dy}\right| f_x(x)$$

  where $\left|\frac{dx}{dy}\right|$ is known as the Jacobian.

Example. If $x \sim$Uniform$(0, T)$ and $y = 1/x$:

$$\begin{aligned} f_x(x) &= 1/T, \\ x &= 1/y, \\ \frac{dx}{dy} &= -1/y^2, \end{aligned}$$

so the Jacobian is

$$\left|\frac{dx}{dy}\right| = 1/y^2$$

which gives the distribution for y:

$$f_y(y) = \frac{1}{y^2 T}.$$

Law of total variance

We know the law of Y and the law of X conditional on Y. We want to compute the variance of X.

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= E[\underbrace{E[X^2 \,|\, Y]}_{a}] - E[E[X \,|\, Y]]^2 \end{aligned}$$

Use the definition of variance to get

$$a = E[X^2 \,|\, Y] = Var[X \,|\, Y] + E[X \,|\, Y]^2$$

and obtain

$$\begin{aligned} Var[X] &= E[a] - E[E[X \,|\, Y]]^2 \\ &= E[Var[X \,|\, Y] + E[X \,|\, Y]^2] - E[E[X \,|\, Y]]^2 \\ &= E[Var[X \,|\, Y]] + \left(E[E[X \,|\, Y]^2] - E[E[X \,|\, Y]]^2\right) \\ &= E[Var[X \,|\, Y]] + Var[E[X \,|\, Y]] \end{aligned}$$

Laplace transform of a gamma random variable

Let x be $\gamma(a, b)$. Its density function is

$$f_x(x) = e^{-x/b} x^{a-1} b^{-a} \Gamma(a)^{-1}$$

where $\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx$. The Laplace transform is derived as follows

$$\begin{aligned} \varphi_x(u) &= E[e^{ux}] \\ &= \int_0^\infty e^{ux} f_x(x) dx \\ &= \int_0^\infty e^{ux} e^{-x/b} x^{a-1} b^{-a} \Gamma(a)^{-1} dx \\ &= b^{-a} \Gamma(a)^{-1} \int_0^\infty e^{ux - x/b} x^{a-1} dx \\ &= b^{-a} \Gamma(a)^{-1} (1/b - u)^{-a} \int_0^\infty e^{-x} x^{a-1} dx \\ &= b^{-a} \Gamma(a)^{-1} (1 - ub)^{-a} b^a \Gamma(a) \\ &= (1 - ub)^{-a} \end{aligned}$$

Laplace transform of a Poisson random variable

Let x be $\mathcal{P}(a)$. Its probability function is

$$p_x(k) = e^{-a} a^k (k!)^{-1}.$$

The Laplace transform is derived as follows

$$\begin{aligned}
\varphi_x(u) &= E[e^{ux}] \\
&= \sum_{k=0}^{\infty} e^{uk} p_x(k) \\
&= e^{-a} \sum_{k=0}^{\infty} e^{uk} a^k (k!)^{-1} \\
&= e^{-a} \sum_{k=0}^{\infty} (e^u a)^k (k!)^{-1} \\
&= e^{-a} e^{ae^u} \\
&= e^{a(e^u - 1)}
\end{aligned}$$