Fixed Income and Credit Risk : solutions for exercise sheet n° 05

Fall Semester 2012

Professor	Assistant	Program
Fulvio Pegoraro	Roberto Marfè	MSc. Finance

Exercise N° 01 [Exponential-affine ZCB Pricing Formula].

Given that $M_{t,t+1}$ is exponential-affine in ε_{t+1} (i.e. x_{t+1}) and that the conditional Laplace transform of x_{t+1} is exponential-affine in the conditioning variable (X_t) we suggest that the ZCB pricing formula at date t be an exponential-affine function of X_t and then "we check if it works". We proceed in the following way:

a) We suggest $B(t,h) = \exp(C'_h\, X_t + D_h)$ and we (equivalently) rewrite the pricing formula in terms of the payoff $B(t+1,h-1) = \exp(C'_{h-1}\, X_{t+1} + D_{h-1})$:

$$\begin{aligned} B(t,h) &= \exp(C'_h\, X_t + D_h) \\ &= E_t[M_{t,t+1} \cdots M_{t+H-1,t+H}] \\ &= E_t[M_{t,t+1} B(t+1,h-1)] \\ &= E_t\left[\exp\left(-\beta - \alpha'\, X_t + \Gamma'_t\, \varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma'_t\Gamma_t\right) \exp(C'_{h-1}\, X_{t+1} + D_{h-1})\right] , \end{aligned}$$

b) we do the algebra (calculating the conditional Laplace transform) obtaining:

$$\begin{aligned} & B(t,h) \\ =\ & \exp({C_h}' X_t + D_h) \\ =\ & \exp\left[-\beta - \alpha' X_t - \tfrac{1}{2}\Gamma'_t\Gamma_t + D_{h-1}\right] \times E_t[\exp\left(\Gamma'_t\varepsilon_{t+1} + C'_{h-1}X_{t+1}\right)] \\ =\ & \exp\left[-\beta - \alpha' X_t - \tfrac{1}{2}\Gamma'_t\Gamma_t + D_{h-1} + C'_{h-1}\widetilde{\Phi} X_t + C'_{1,h-1}\nu\right] \\ & \quad \times E_t[\exp\left(\Gamma_t + \Sigma' C_{1,h-1}\right)'\varepsilon_{t+1})] \\ =\ & \exp\Bigg[\left(-\alpha + \widetilde{\Phi}' C_{h-1} + (\Sigma\gamma)' C_{1,h-1}\right)' X_t \\ & \quad + \left(-\beta + C'_{1,h-1}(\nu + \Sigma\widetilde{\Gamma}) + \tfrac{1}{2} C'_{1,h-1}(\Sigma\Sigma')C_{1,h-1} + D_{h-1}\right)\Bigg] , \end{aligned}$$

c) and by identifying the coefficients we find the recursive relations for C_h and D_h characterizing the pricing formula $B(t,h) = \exp(C_h' X_t + D_h)$.

Now, the last elements we need to completely determine the pricing formula are the starting conditions for C_h and D_h. We proceed as follows:

given that, by definition of ZCB, we have $B(t,0) = 1$, then

$$\exp(C_0' X_t + D_0) = 1 \iff (C_0' X_t + D_0) = 0 \ \ \forall \ X_t \iff C_0 = 0 \,, \ \ D_0 = 0 \,.$$

We can also equivalently write:

given that, by definition of ZCB, we have $B(t,1) = \exp(-\, r_t)$, then

$$\exp(C_1' X_t + D_1) = \exp(-\, r_t) \iff (C_1' X_t + D_1) = -r_t \ \ \forall \ X_t \iff C_1 = -\,\alpha \,, \ \ D_1 = -\,\beta \,.$$

Exercise N° 02 [A different derivation of the Gaussian ATSM - Scalar case].

We know that, under the absence of arbitrage opportunities, there exists a risk-neutral probability measure $\mathbb{Q}$ such that the price at date t for a ZCB of residual maturity h is given by $B(t,h) = E_t^{\mathbb{Q}}[\exp(-r_t - \ldots - r_{t+h-1})]$. This means that, the yield-to-maturity formula is given by $R(t,h) = -\frac{1}{h} \ln E_t^{\mathbb{Q}}[\exp(-r_t - \ldots - r_{t+h-1})]$.

i) We want to determine the yield-to-maturity formula $R(t,h)$ in the case where the scalar factor (x_t) follows the following Gaussian AR(1) process:

$$x_{t+1} \ \ = \ \ \nu^* \, + \, \varphi^* x_t \, + \sigma^* \eta_{t+1} \,, \ \ \eta_{t+1} \sim \mathcal{N}(0,1) \ \ (\text{under } \mathbb{Q}) \,,$$

and the short rate process is assumed to be $r_t = \beta + \alpha x_t$. We follows the same steps we have seen during Lecture 4, that is, first we determine the ZCB pricing formula $B(t,h)$ guessing an exponential-affine (in x_t) functional form and then we determine the associated interest rates formula $R(t,h)$.

- First step :

$$\begin{aligned}
B(t,h) &= \exp(c_h \, x_t + d_h) \\
&= E_t^{\mathbb{Q}}[\exp(-r_t - \ldots - r_{t+h-1})] \\
&= E_t^{\mathbb{Q}}[\exp(-r_t) B(t+1, h-1)] \\
&= E_t^{\mathbb{Q}} \left[\exp(-\beta - \alpha x_t) \exp(c_{h-1} \, x_{t+1} + d_{h-1})\right] \,, \\
&= \exp(-\beta - \alpha x_t + d_{h-1}) \, E_t^{\mathbb{Q}} \left[c_{h-1} \, x_{t+1}\right] \,, \\
&= \exp(-\beta - \alpha x_t + d_{h-1} + c_{h-1}\nu^* + c_{h-1}\varphi^* x_t) \, E_t^{\mathbb{Q}} \left[c_{h-1} \, \sigma^* \eta_{t+1}\right] \,, \\
&= \exp\left(-\beta - \alpha x_t + d_{h-1} + c_{h-1}\nu^* + c_{h-1}\varphi^* x_t + \tfrac{1}{2} c_{h-1}^2 (\sigma^*)^2\right) \\
&= \exp\left[(-\alpha + c_{h-1}\varphi^*) x_t + (-\beta + c_{h-1}\nu^* + \tfrac{1}{2} c_{h-1}^2 (\sigma^*)^2 + d_{h-1})\right] \,,
\end{aligned}$$

and therefore, by identification, we find that $B(t,h) = \exp(c_h\, x_t + d_h)$ where (c_h, d_h) are given by:

$$\begin{cases} c_h &= -\alpha + \varphi^* c_{h-1}\,, \\ d_h &= -\beta + c_{h-1}\nu^* + \frac{1}{2} c_{h-1}^2 (\sigma^*)^2 + d_{h-1}\,, \end{cases}$$

with $c_0 = 0$ and $d_0 = 0$ given that $B(t,t) = 1$.

- Second step : the yield-to-maturity formula is clearly given by $R(t,h) = -\frac{1}{h}[c_h x_t + d_h]$ and therefore we have found the same ZCB price and yield-to-maturity formulas as in the case presented during Lecture 4.

  Indeed, the functional form are the same and the recursive equations are the same. The only difference is that the methodology specifying the historical dynamics and the SDF $M_{t,t+1}$ allows to decompose ν^* and φ^* in terms of historical and risk-premia parameters : $\nu^* = \nu + \sigma\gamma_o$ and $\varphi^* = \varphi + \sigma\gamma$ $(\sigma = \sigma^*)$. Indeed, giving a value (directly) to $(\nu^*, \varphi^*, \sigma^*)$ or a value to $(\nu, \varphi, \sigma, \gamma_o, \gamma)$ specify exactly the same recursive equations and therefore the prices are the same given that the functional form is the same.

Observe that, the assumption $r_t = \beta + \alpha x_t$ is made to automatically guarantee that under $\mathbb{Q}$ discounted asset prices are martingales, that is, to automatically satisfy the condition $B(t,1) = E_t^{\mathbb{Q}}[\exp(-r_t)] = \exp(-r_t)$ (r_t is known in t). Indeed, from the formula $B(t,h)$ with $h = 1$ we find $B(t,1) = \exp(c_1 x_t + d_h) = \exp(-\alpha x_t - \beta)$ and therefore $B(t,1) = E_t^{\mathbb{Q}}[\exp(-r_t)] = \exp(-r_t)$ is satisfied if and only if $r_t = \beta + \alpha x_t$.

This means that, before starting to calculate $B(t,h) = E_t^{\mathbb{Q}}[\exp(-r_t - \ldots - r_{t+h-1})]$ we have to impose that $\mathbb{Q}$ be an risk-neutral probability measure that is a probability measure such that any discounted asset price (discounted by the short rate sequence) which in the information of the investor is a martingale.

ii) If we consider the case in which $x_t = r_t$ we have:

$$r_{t+1} \;=\; \nu^* + \varphi^* r_t + \sigma^* \eta_{t+1}\,, \;\; \eta_{t+1} \sim \mathcal{N}(0,1) \;\; (\text{under } \mathbb{Q})\,,$$

and following the same steps as before we find $B(t,h) = \exp(c_h x_t + d_h)$ and $R(t,h) = -\frac{1}{h}[c_h x_t + d_h]$ with:

$$\begin{cases} c_h &= -1 + \varphi^* c_{h-1}\,, \\ d_h &= c_{h-1}\nu^* + \frac{1}{2} c_{h-1}^2 (\sigma^*)^2 + d_{h-1}\,, \end{cases}$$

with $c_0 = 0$ and $d_0 = 0$. Observe that, again, these formulas are the same we have presented during Lecture 4 (and 5) with $\alpha = 1$ and $\beta = 0$. Observe also that the condition $B(t,1) = E_t^{\mathbb{Q}}[\exp(-r_t)] = \exp(-r_t)$ is automatically satisfied.

Exercise N° 03.

(i) Let us assume that $M_{t,t+1}(\underline{w}_{t+1})$ has an exponential-affine form :

$$M_{t,t+1} = \exp\left[\Gamma_t(\underline{w}_t)' w_{t+1} + \beta_t(\underline{w}_t)\right] ,$$

where Γ_t denotes the market price of factor's risk function. Since $\exp(-r_{t+1}) = E_t(M_{t,t+1}) = \exp\left[\psi_t(\Gamma_t \mid \underline{w}_t) + \beta_t\right]$, the SDF can also be written :

$$M_{t,t+1} = \exp\left[-r_{t+1}(\underline{w}_t) + \Gamma_t'(\underline{w}_t) w_{t+1} - \psi_t(\Gamma_t|\underline{w}_t)\right]$$

and therefore, this specification (function of w_{t+1} instead of its noise) automatically satisfy the condition $\exp(-r_{t+1}) = E_t(M_{t,t+1})$.

(ii) The joint historical distribution of $\underline{w}_t$, denoted by $\mathbb{P}$, is defined by the conditional distribution of w_{t+1} given $\underline{w}_t$, characterized either by the p.d.f. $f_t(w_{t+1}|\underline{w}_t)$ or the Laplace transform $\varphi_t(u|\underline{w}_t)$, or the Log-Laplace transform $\psi_t(u|\underline{w}_t)$. The Risk-Neutral (R.N.) dynamics is another joint distribution of $\underline{w}_t$, denoted by $\mathbb{Q}$, defined by the conditional p.d.f., with respect to the corresponding conditional historical probability, given by :

$$\begin{aligned} \frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} := d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) &= \frac{M_{t,t+1}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}(\underline{w}_{t+1})\right]} \\ &= \exp(r_{t+1}) M_{t,t+1}(\underline{w}_{t+1}). \end{aligned}$$

So, the R.N. conditional p.d.f. (with respect to the same measure as the corresponding conditional historical probability) is :

$$f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t) d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t),$$

and the conditional p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$\frac{d\mathbb{P}_{t,t+1}}{d\mathbb{Q}_{t,t+1}} := d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \frac{1}{d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t)}.$$

When the SDF is exponential-affine, we have convenient additional results:

$$\begin{aligned} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) &= \frac{\exp(\Gamma_t' w_{t+1} + \beta_t)}{E_t \exp(\Gamma_t' w_{t+1} + \beta_t)} \\ &= \exp\left[\Gamma_t' w_{t+1} - \psi_t(\Gamma_t)\right], \end{aligned}$$

so $d_t^{\mathbb{Q}}$ is also exponential-affine. The conditional R.N. Laplace transform of the factor w_{t+1}, given $\underline{w}_t$, is :

$$\begin{aligned} \varphi_t^{\mathbb{Q}}(u|\underline{w}_t) &= E_t^{\mathbb{Q}}\left[\exp(u' w_{t+1})\right] \\ &= E_t \exp\left[(u + \Gamma_t)' w_{t+1} - \psi_t(\Gamma_t)\right] \\ &= \frac{\varphi_t(u + \Gamma_t)}{\varphi_t(\Gamma_t)} \end{aligned}$$

and, consequently, the associated conditional R.N. Log-Laplace transform is :

$$\psi_t^{\mathbb{Q}}(u) = \psi_t(u + \Gamma_t) - \psi_t(\Gamma_t)\,.$$

Conversely, we get :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \exp\left[-\Gamma_t' w_{t+1} + \psi_t(\Gamma_t)\right]$$

and, taking $u = -\Gamma_t$ in $\psi_t^{\mathbb{Q}}(u)$, we can write :

$$\psi_t^{\mathbb{Q}}(-\Gamma_t) = -\psi_t(\Gamma_t)$$

and, replacing u by $u - \Gamma_t$, we obtain :

$$\psi_t(u) = \psi_t^{\mathbb{Q}}(u - \Gamma_t) - \psi_t^{\mathbb{Q}}(-\Gamma_t).$$

We also have :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \exp\left[-\Gamma_t' w_{t+1} - \psi_t^{\mathbb{Q}}(-\Gamma_t)\right]\,,$$

$$d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = \exp\left[\Gamma_t' w_{t+1} + \psi_t^{\mathbb{Q}}(-\Gamma_t)\right]\,.$$

Exercise N° 04 [Exercise N° 03, continued].

(i) We have a Gaussian AR(1) latent process x_t such that:

$$x_{t+1} \;=\; \nu^* + \varphi^* x_t + \sigma^* \eta_{t+1}\,,\;\; \eta_{t+1} \sim \mathcal{N}(0,1) \;\text{(under } \mathbb{Q})\,.$$

We know that the conditional risk-neutral log-Laplace transform of η_t is given by $\psi_t^{\mathbb{Q}}(-\Gamma_t) = \text{Log}E_t[\exp(-\Gamma_t\,\eta_{t+1})] = \exp(\Gamma_t^2/2)$, and therefore the one-period SDF $M_{t,t+1} = M_{t,t+1}(\eta_{t+1})$ is given by:

$$\begin{aligned} M_{t,t+1} &= \exp\left[-\beta - \alpha x_t + \Gamma_t \eta_{t+1} + \tfrac{1}{2}\Gamma_t^2\right]\,,\;\; \text{(SDF)} \\ \Gamma_t &= \Gamma(x_t) = (\gamma_o + \gamma x_t)\,. \end{aligned}$$

(ii) In order to determine the historical dynamics of x_t let us work with the conditional Laplace transform, and let us remember that for any risk-neutral probability measure $\mathbb{Q}$ (equivalent to $\mathbb{P}$) we have:

$$\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{M_{t,t+1}}{E_t[M_{t,t+1}]}\,,\;\text{ and }\; \frac{d\mathbb{P}_{t,t+1}}{d\mathbb{Q}_{t,t+1}} = \frac{E_t[M_{t,t+1}]}{M_{t,t+1}}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{M_{0,1}\ldots M_{T-1,T}}{E_0[M_{0,1}]\ldots E_{T-1}[M_{T-1,T}]}\,,\;\text{ and }\; \frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{E_0[M_{0,1}]\ldots E_{T-1}[M_{T-1,T}]}{M_{0,1}\ldots M_{T-1,T}}\,.$$

Now, we have that the historical Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$E_t[\exp(ux_{t+1})] = E_t^{\mathbb{Q}}\left[\frac{E_t[M_{t,t+1}]}{M_{t,t+1}}\exp(ux_{t+1})\right]$$

$$= E_t^{\mathbb{Q}}\left[\exp\left(-(\gamma_o+\gamma x_t)\,\eta_{t+1} - \tfrac{1}{2}(\gamma_o+\gamma x_t)^2 + ux_{t+1}\right)\right]$$

$$= \exp\left[u(\nu^*+\varphi^* x_t) - \tfrac{1}{2}(\gamma_o+\gamma x_t)^2\right] \times E_t^{\mathbb{Q}}[\exp(-\gamma_o-\gamma x_t+u\sigma^*)\eta_{t+1}]$$

$$= \exp\left[u[(\nu^*-\sigma^*\gamma_o)+(\varphi^*-\sigma^*\gamma)x_t] + \tfrac{1}{2}u^2(\sigma^*)^2\right]$$

$$= \exp\left[u(\nu+\varphi x_t) + \tfrac{1}{2}u^2\sigma^2\right],$$

This means that, under $\mathbb{P}$, (x_t) follows a Gaussian AR(1) process:

$$x_{t+1} \;=\; \nu + \varphi x_t + \sigma\varepsilon_{t+1}\,,\;\; \varepsilon_{t+1} \sim \mathcal{N}(0,1)\;\text{(under }\mathbb{P}),$$

where $\nu = (\nu^* - \sigma^*\gamma_o)$, $\varphi = (\varphi^* - \sigma^*\gamma)$ and $\sigma^* = \sigma$. We also find that $\eta_{t+1} = \varepsilon_{t+1} - \Gamma_t$.

Even if the ZCB pricing formula can be determined simply making an assumption about the risk-neutral factor dynamics, the specification of the historical dynamics becomes essential (for instance) if we need to forecast future interest rates:

$$E_t[R(t+k,h)] = -\frac{c_h}{h}E_t[x_{t+k}] - \frac{d_h}{h}$$

(forecasts are under $\mathbb{P}$!). Observe that the affine nature of the yield-to-maturity formula makes the forecast easy to be implemented.

Exercise N° 05 [No-arbitrage restrictions for the short and long rate].

We have a bivariate Gaussian VAR(1) ATSM given by:

$$x_{t+1} = \nu + \Phi x_t + \Sigma\varepsilon_{t+1}\,,\;\; \varepsilon_{t+1} \sim \mathcal{N}(0,I_2)\;\text{(under }\mathbb{P})$$

$$M_{t,t+1} = \exp\left[-\beta - \alpha' x_t + \Gamma_t'\varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t'\Gamma_t\right]\,,\;\;\text{(SDF)}$$

$$\Gamma_t = \Gamma(x_t) = (\gamma_o + \gamma\, x_t)\,,$$

$$R(t,t+h) = -\frac{C_h{}'}{h}x_t - \frac{D_h}{h}\,,$$

$$C_h = -\alpha + (\Phi+\Sigma\gamma)'C_{h-1} = -\alpha + \Phi^{*\prime}C_{h-1}\,,$$

$$D_h = -\beta + C_{h-1}'(\nu+\Sigma\gamma_o) + \tfrac{1}{2}C_{h-1}'(\Sigma\Sigma')C_{h-1} + D_{h-1}\,,$$

$$C_0 = 0, D_0 = 0\,,$$

where $x_t = (r_t, R_t)'$, with $r_t = R(t, t+1)$ the yield with the shortest maturity in our data base (it is the short rate) and $R_t = R(t, t+H)$ the long rate, i.e. the yield with the longest maturity in our data base.
I have to impose no-arbitrage restrictions on both components of the factor (x_t) given that they contains yields at different maturities.
First, I have to impose that $R(t, t+1) = r_t$: this condition generates the no-arbitrage restriction $R(t, t+1) = \beta + \alpha' x_t = \beta + \alpha_1 r_t + \alpha_2 R_t = r_t$. Clearly, $r_t = \beta + \alpha' x_t = \beta + \alpha_1 r_t + \alpha_2 R_t$ if and only if $\beta = 0$, $\alpha_1 = 1$ and $\alpha_2 = 0$. These conditions are equivalent to $C_1 = -(1, 0)$ and $D_1 = 0$.
Second, let us denote by H the longest maturity in our data base. I have to impose that $R(t, t+H) = R_t$ for any t. In this case we have:

$$-\frac{1}{H}[C_{1,H}\, r_t + C_{2,H}\, R_t + D_H] = R_t$$

$$\Leftrightarrow C_{1,H}\, r_t + C_{2,H}\, R_t + D_H = -H R_t$$

$$\Leftrightarrow C_{1,H} = 0\,,\;\; C_{2,H} = -H\,,\;\; D_H = 0\,,$$

that is $C_H = -H\,(0, 1)'$ and $D_H = 0$.

Exercise N° 06 [Conditional distribution of yields when the factor is Gaussian AR(p)].

We have a Gaussian AR(p) Factor-Based term structure model in which the factor (x_t) is assumed latent. For a fixed time to maturity h, the process $R = [\,R(t,h),\; 0 \le t < T\,]$ is an ARMA(p, $p-1$) process of the following type :

$$\Psi(L)R(t,h) \;\; = \;\; \sigma \mathbf{C}_h(L)\varepsilon_t + \mathbf{C}_h(1)\nu + \Psi(1)\delta_h\,,$$

where $\mathbf{C}_h(L) = -(c_{1,h} + c_{2,h}L + \ldots + c_{p,h}L^{p-1})/h$ is a polynomial of degree $(p-1)$ in the lag operator L, $\delta_h = -(d_h/h)$, and where the AR polynomial, applying to t, is given by $\Psi(L) = (1 - \varphi_1 L - \ldots \varphi_p L^p)$.

Indeed, we can write the yield-to-maturity formula $R(t,h) = -\frac{1}{h}[c_h' X_t + d_h]$ in the following way:

$$R(t,h) \;\; = \;\; \mathbf{C}_h(L)x_t + \delta_h\,,$$

where $\mathbf{C}_h(L) = -(c_{1,h} + c_{2,h}L + \ldots + c_{p,h}L^{p-1})/h$ is the $(p-1)^{th}$ degree polynomial in the backward shift operator L, and where $\delta_h = -(d_h/h)$.

Now, if we apply on the right-hand and left-hand side of this relation the operator $\Psi(L) = (1 - \varphi_1 L - \ldots \varphi_p L^p)$ operating on t, we can write :

$$\begin{aligned}\Psi(L)R(t,h) \;\; &= \;\; \mathbf{C}_h(L)\Psi(L)x_t + \Psi(1)\delta_h \\ &= \;\; \mathbf{C}_h(L)[\,\nu + \sigma\varepsilon_t] + \Psi(1)\delta_h = \sigma\mathbf{C}_h(L)\varepsilon_t + \mathbf{C}_h(1)\nu + \Psi(1)\delta_h\,,\end{aligned}$$

showing that $R = [\,R(t,h),\; 0 \le t < T\,]$ is an ARMA(p, $p-1$) process. Observe that the AR polynomial is independent of h, while the MA polynomial is not. Moreover, we also highlight the fact that, when $p = 1$, any yield follow a Gaussian AR(1) process.

Exercise N° 07 [Conditional p.d.f. of yields when the factor is Gaussian VAR(1)].

We have Gaussian VAR(1) Factor-Based term structure model in which the latent factor (x_t) is K-dimensional. Let us consider, at date t, K yields that we organize in the vector $R_K(t) = [R(t,h_1),\ldots,R(t,h_K)]'$. Now, the affine relation between this vector of yields and the factor x_t can be written in the following way:

$$R_K(t) = \mathcal{C}_K\, x_t + \mathcal{D}_K\,,$$

$$\text{where } \mathcal{C}_K = \begin{bmatrix} -\dfrac{c_{1,h_1}}{h_1} & \cdots & -\dfrac{c_{K,h_1}}{h_1} \\ \vdots & \ddots & \vdots \\ -\dfrac{c_{1,h_K}}{h_K} & \cdots & -\dfrac{c_{K,h_K}}{h_K} \end{bmatrix}, \quad \text{and} \quad \mathcal{D}_K = \begin{bmatrix} -\dfrac{d_{h_1}}{h_1} \\ \vdots \\ -\dfrac{d_{h_K}}{h_K} \end{bmatrix}$$

which is a system of K equations in K unknowns (the scalar variables in x_t). Given the observed yields $R_K(t)$, we can easily solve for x_t and write:

$$x_t = \mathcal{C}_K^{-1}\,[R_K(t) - \mathcal{D}_K]\,.$$

Now, given that the conditional p.d.f. $f(x_{t+1}\,|\,x_t)$ is known (it is the p.d.f. of K-dimensional conditional Gaussian process with conditional mean $E_t[x_{t+1}] = \nu + \Phi x_t$ and conditional variance $V_t[x_{t+1}] = \Omega$), we have that the p.d.f. $f(R_K(t+1)\,|\,R_K(t))$ follows directly from $f(x_{t+1}\,|\,x_t)$ and involves the Jacobian of the transformation from $R_K(t)$ to x_t.
Since the transformation (forgetting t for a while) is $x[R_K] = \mathcal{C}_K^{-1}\,[R_K - \mathcal{D}_K]$, its Jacobian is:

$$J = \det\left(\frac{d\,x[R_K]}{dR_K}\right) = \det(\mathcal{C}_K^{-1}) = \frac{1}{\det(\mathcal{C}_K)}$$

which implies that the historical conditional p.d.f. $f(R_K(t+1)\,|\,R_K(t))$ of the yields is given by:

$$f(R_K(t+1)\,|\,R_K(t)) = \frac{1}{\det(\mathcal{C}_K)}\, f(x_{t+1}\,|\,x_t)\,.$$

Given the set of observations at times $\{t_1,\ldots,t_n\}$, the log-Likelihood function is given by:

$$\mathcal{L} = \sum_{i=1}^{n} \log f(R_K(t_i)\,|\,R_K(t_{i-1}))\,,$$

$$\text{assuming } f(R_K(t_1)\,|\,R_K(t_0)) = f(R_K(t_1))\,, \text{ i.e., the marginal density}\,.$$

Observe that this methodology is applied to the case $p = 1$, and it estimates model parameters using a number of yields that has to be equal to the dimension of the factor. Chen and Scott (1993) tackle this problem assuming that additional yields are observed with errors. Let us assume that $M-K$ additional yield are measured with errors, besides the K observed without errors:

$$R_t^{M-K} = \mathcal{C}_{M-K}\, x_t + \mathcal{D}_{M-K} + \eta_t\,,$$

$$R_t^{M-K} = [R(t,t+h_{K+1}),\ldots,R(t,t+h_M)]'\,,$$

where the conditional distribution of the measurement errors (η_t) is known and given by $h(\eta_t \,|\, \eta_{t-1})$. Moreover, $\eta_t \perp R_t^{M-K}$, $\eta_t \perp R_t^{K}$. In order to determine the log-Likelihood function associated to the vector of yields $R_t^M = (R_t^{K\prime}, R_t^{M-K\prime})'$ we have to simply specify the associated conditional p.d.f. taking into account the above mentioned assumptions:

$$\begin{aligned} f(R_t^M \,|\, R_{t-1}^M) &= f(R_t^M \,|\, R_{t-1}^K, \eta_{t-1}) \\ &= f(R_t^K, \eta_t \,|\, R_{t-1}^K, \eta_{t-1}) \\ &= f(R_t^K \,|\, \eta_t, R_{t-1}^K, \eta_{t-1})\, f(\eta_t \,|\, R_{t-1}^K, \eta_{t-1}) \\ &= f(R_t^K \,|\, R_{t-1}^K)\, f(\eta_t \,|\, \eta_{t-1})\,. \end{aligned}$$

This means that $\log f(R_t^M \,|\, R_{t-1}^M) = \log f(R_t^K \,|\, R_{t-1}^K) + \log f(\eta_t \,|\, \eta_{t-1})$ and, thus:

$$\mathcal{L}^*(\theta) = \mathcal{L}(\theta) + \sum_{t=1}^{T} \log h(\eta_t \,|\, \eta_{t-1})\,,$$

$$\text{assuming } \; h(\eta_1 \,|\, \eta_0) = h(\eta_1)\,, \; \text{ i.e., the marginal density}\,.$$

Exercise N° 08 [Working with the Non-centered Gamma Distribution].

Given that the non-centered Gamma random variable Y can be represented in the following way:

$$\left\{ \begin{array}{l} \dfrac{Y}{\mu} \,|\, Z \sim \gamma(\nu + Z, 1)\,, \;\; \nu > 0\,, \\[2ex] Z \sim \mathcal{P}(\beta)\,, \;\; \beta > 0\,, \mu > 0\,, \end{array} \right. \iff \left\{ \begin{array}{l} Y \,|\, Z \sim \gamma(\nu + Z, \mu)\,, \;\; \nu > 0\,, \\[2ex] Z \sim \mathcal{P}(\beta)\,, \;\; \beta > 0\,, \mu > 0\,, \end{array} \right.$$

then, we can easily determine its mean, variance and Laplace transform. Indeed:

i) Given that $Y \,|\, Z \sim \gamma(\nu + Z, \mu)$ and that $E(Y) = E[E(Y \,|\, Z)]$ we easily find that $E(Y) = E[\mu(\nu + Z)] = \mu\,\nu + \mu E(Z) = \mu\,\nu + \mu\,\beta$.

ii) Given that $V(Y) = E[V(Y \,|\, Z)] + V[E(Y \,|\, Z)]$, again we easily find:

$$\begin{aligned} V(Y) &= E[V(Y \,|\, Z)] + V[E(Y \,|\, Z)] \\ &= E[\mu^2(\nu + Z)] + V[\mu(\nu + Z)] \\ &= \nu\,\mu^2 + 2\mu^2\beta\,. \end{aligned}$$

iii) With regard to the Laplace transform, we have:

$$\begin{aligned}
\varphi(u) &= E[\exp(u\,Y)] = E_Z\left[E_{Y\,|\,Z}(\exp(uY)\,|\,Z)\right] \\
&= E_Z\left[\left(\frac{1}{1-\mu u}\right)^{\nu+Z}\right] \\
&= \left(\frac{1}{1-\mu u}\right)^{\nu} E_Z\left[\left(\frac{1}{1-\mu u}\right)^{Z}\right] \\
&= \exp[-\nu\log(1-\mu u)]\, E_Z[\exp(-Z\,\log(1-\mu\,u))] \\
&= \exp[-\nu\log(1-\mu u)]\,\exp\left[\beta\,\frac{u\mu}{1-u\mu}\right] \\
&= \exp\left[-\nu\log(1-\mu u)+\beta\,\frac{u\mu}{1-u\mu}\right].
\end{aligned}$$

Exercise N° 09 [Working with the ARG(1) Process].

We have an ARG(1) process (x_t) defined as:

$$\frac{x_{t+1}}{\mu}|z_{t+1} \sim \gamma(\nu+z_{t+1},1)\,,\;\; \nu>0\,,$$

$$z_{t+1}|x_t \sim \mathcal{P}(\rho x_t/\mu)\,,\;\; \rho>0\,,\mu>0\,,\rho=\beta\,\mu$$

then, following the previous exercise, we can write:

i)

$$\begin{aligned}
E(x_{t+1}\,|\,x_t) &= E[E(x_{t+1}\,|\,z_{t+1}\,,x_t)\,|\,x_t] = E[(\mu(\nu+z_{t+1}))\,|\,x_t] \\
&= \nu\,\mu+\mu E(z_{t+1}\,|\,x_t) = \nu\,\mu+\mu\,\beta\,x_t = \nu\,\mu+\rho\,x_t\,.
\end{aligned}$$

ii)

$$\begin{aligned}
V(x_{t+1}\,|\,x_t) &= E[V(x_{t+1}\,|\,z_{t+1}\,,x_t)\,|\,x_t]+V[E(x_{t+1}\,|\,z_{t+1}\,,x_t)\,|\,x_t] \\
&= E[\mu^2(\nu+z_{t+1})\,|\,x_t]+V[\mu(\nu+z_{t+1})\,|\,x_t] \\
&= \mu^2\,\nu+\mu^2\,E(z_{t+1}\,|\,x_t)+\mu^2\,V(z_{t+1}\,|\,x_t) \\
&= \mu^2\,\nu+2\mu^2\,\beta\,x_t = \mu^2\,\nu+2\mu\,\rho\,x_t\,.
\end{aligned}$$

iii) Under the assumption of stationarity ($0<\rho<1$), we have $E(x_t)=E(x)$ and $V(x_t)=V(x)$

for all t. Then, we can write:

$$E(x_t) = E[E(x_t \,|\, x_{t-1})] = E[\nu\,\mu + \rho\, x_{t-1}] = \nu\,\mu + \rho\, E(x_{t-1})$$

$$\Rightarrow \text{ under stationarity } E(x) = \nu\,\mu + \rho\, E(x)$$

$$\Rightarrow E(x) = \frac{\mu\,\nu}{1-\rho},$$

and therefore the marginal mean is, for every t, given by $E(x_t) = \dfrac{\mu\,\nu}{1-\rho}$.

iv) In the same, we can work for the marginal variance:

$$\begin{aligned} V(x_t) &= E[V(x_t \,|\, x_{t-1})] + V[E(x_t \,|\, x_{t-1})] \\ &= E[\mu^2\,\nu + 2\mu\,\rho\, x_{t-1}] + V[\nu\,\mu + \rho\, x_{t-1}] \\ &= \mu^2\,\nu + 2\mu\,\rho\, E(x_{t-1}) + \rho^2 V(x_{t-1}) \\ &= \mu^2\,\nu + 2\mu\,\rho\, \frac{\mu\,\nu}{1-\rho} + \rho^2 V(x_{t-1}) \end{aligned}$$

and, therefore, under stationarity we can write:

$$\begin{aligned} V(x) &= \frac{1}{1-\rho^2}\left[\mu^2\,\nu + 2\mu\,\rho\,\frac{\mu\,\nu}{1-\rho}\right] \\ &= \frac{1}{1-\rho^2}\,\frac{\mu^2\,\nu(1+\rho)}{(1-\rho)} = \frac{\mu^2\,\nu}{(1-\rho)^2}\,. \end{aligned}$$

v) With regard to the conditional Laplace transform, we have:

$$\begin{aligned} \varphi_t(u) &= E[\exp(u\, x_{t+1}) \,|\, x_t] = E\left[E(\exp(u\, x_{t+1}) \,|\, z_{t+1}, x_t) \,|\, x_t\right] \\ &= E\left[\left(\frac{1}{1-\mu u}\right)^{\nu + z_{t+1}} \,|\, x_t\right] \\ &= \left(\frac{1}{1-\mu u}\right)^{\nu} E\left[\left(\frac{1}{1-\mu u}\right)^{z_{t+1}} \,|\, x_t\right] \\ &= \exp[-\nu \log(1-\mu u)]\, E[\exp(-z_{t+1}\, \log(1-\mu\, u)) \,|\, x_t] \\ &= \exp[-\nu \log(1-\mu u)]\, \exp\left[\beta\, x_t\, \frac{u\mu}{1-u\mu}\right] \\ &= \exp\left[-\nu \log(1-\mu u) + \frac{\rho\, u}{1-u\mu}\, x_t\right]. \end{aligned}$$

vi) Our ARG(1) process, with $E(x_{t+1}\,|\,x_t) = \nu\,\mu + \rho\,x_t$ and $V(x_{t+1}\,|\,x_t) = \mu^2\,\nu + 2\mu\,\rho\,x_t$, can be represented by the following weak positive AR(1) model:

$$x_{t+1} = \nu\,\mu + \rho\,x_t + \varepsilon_{t+1}$$

where, by construction, $E(\varepsilon_{t+1}\,|\,\underline{\varepsilon_t}) = E(x_{t+1} - \nu\,\mu - \rho\,x_t\,|\,\underline{x_t}) = E(x_{t+1}\,|\,\underline{x_t}) - \nu\,\mu - \rho\,x_t = 0$ and therefore $E(\varepsilon_{t+1}) = E[E(\varepsilon_{t+1}\,|\,\underline{\varepsilon_t})] = 0$.

With regard to the variance of the noise, we have $V(\varepsilon_{t+1}\,|\,\underline{\varepsilon_t}) = V(x_{t+1} - \nu\,\mu - \rho\,x_t\,|\,\underline{x_t}) = V(x_{t+1}\,|\,\underline{x_t}) = \mu^2\,\nu + 2\mu\,\rho\,x_t$. From this result we easily find that:

$$\begin{aligned} V(\varepsilon_{t+1}) &= V[E(\varepsilon_{t+1}\,|\,\underline{\varepsilon_t})] + E[V(\varepsilon_{t+1}\,|\,\underline{\varepsilon_t})] \\ &= E[V(\varepsilon_{t+1}\,|\,\underline{\varepsilon_t})] = E[V(x_{t+1} - \nu\,\mu - \rho\,x_t\,|\,\underline{x_t})] \\ &= E[\mu^2\,\nu + 2\mu\,\rho\,x_t] = \mu^2\,\nu + 2\mu^2\,\rho\,\frac{\nu}{1-\rho} \end{aligned}$$

Exercise N° 10 [The ARG(1) and ARG(p) Positive Affine Yield Curve].

i) We have that the scalar latent factor x_t has an historical dynamics given by the following ARG(1) process:

$$\frac{x_{t+1}}{\mu}\,|\,z_{t+1} \sim \gamma(\nu + z_{t+1}, 1)\,, \quad \nu > 0\,,$$

$$z_{t+1}\,|\,\underline{x_t} \sim \mathcal{P}\left(\frac{\rho x_t}{\mu}\right)\,, \quad \rho = \beta\,\mu\,,$$

that is, $x_{t+1} = \nu\mu + \rho x_t + \varepsilon_{t+1}$ (weak positive AR(1) representation). This means that, under $\mathbb{P}$, the Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$\begin{aligned} E\left[\exp(u x_{t+1})\,|\,\underline{x_t}\right] &= \exp\left[\frac{\rho u}{1-u\mu}\,x_t - \nu\log(1-u\mu)\right]\,, \\ &= \exp\left[a(u;\rho,\mu)\,x_t + b(u;\nu,\mu)\right]\,. \end{aligned}$$

and the Laplace transform of ε_{t+1}, conditionally to $\underline{\varepsilon_t}$, is given by:

$$\begin{aligned} E\left[\exp(u\varepsilon_{t+1})\,|\,\underline{\varepsilon_t}\right] &= E\left\{\exp\left[u(x_{t+1} - \nu\mu - \rho\,x_t)\,|\,\underline{x_t}\right]\right\}\,, \\ &= \exp\left[a(u;\rho,\mu)\,x_t + b(u;\nu,\mu) - u\,(\nu\,\mu + \rho\,x_t)\right]\,, \\ &= \exp\left[(a(u;\rho,\mu) - u\rho)\,x_t + b(u;\nu,\mu) - u\,\nu\,\mu\right]\,. \end{aligned}$$

Given that the SDF is:

$$M_{t,t+1} = \exp\left[-\beta - \alpha x_t + \Gamma_t\,\varepsilon_{t+1}\right]\,\exp\left[-a\left(\Gamma_t;\rho,\mu\right)\,x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\,x_t\right)\right],$$

we can now write the following:

$$
\begin{aligned}
B(t,t+h) &= \exp(c_h\, x_t + d_h) \\
&= E_t[M_{t,t+1} \cdots M_{t+H-1,t+H}] \\
&= E_t[M_{t,t+1} B(t+1,t+h)] \\
&= E_t \left\{ \exp\left[-\beta - \alpha x_t + \Gamma_t\, \varepsilon_{t+1}\right] \exp\left[-a\left(\Gamma_t;\rho,\mu\right)\, x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\, x_t\right)\right] \right. \\
&\qquad\qquad \left. \exp(c_{h-1}\, x_{t+1} + d_{h-1}) \right\} \\
&= \exp\left[-\beta - \alpha x_t - a\left(\Gamma_t;\rho,\mu\right)\, x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\, x_t\right) + d_{h-1}\right] \\
&\qquad\qquad E_t[\exp(\Gamma_t\, \varepsilon_{t+1} + c_{h-1}\, x_{t+1})] \\
&= \exp\left[-\beta - \alpha x_t - a\left(\Gamma_t;\rho,\mu\right)\, x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\, x_t\right) + c_{h-1}\left(\nu\mu + \rho\, x_t\right) + d_{h-1}\right] \\
&\qquad\qquad E_t[\exp((\Gamma_t + c_{h-1})\, \varepsilon_{t+1})] \\
&= \exp\left\{-\beta - \alpha x_t + \left[a\left(\Gamma_t + c_{h-1};\rho,\mu\right) - a\left(\Gamma_t;\rho,\mu\right)\right] x_t + \right. \\
&\qquad\qquad \left. \left[b\left(\Gamma_t + c_{h-1};\nu,\mu\right) - b\left(\Gamma_t;\nu,\mu\right)\right] + d_{h-1} \right\} \\
&= \exp\left\{\left[-\alpha + a\left(c_{h-1};\rho^*,\mu^*\right)\right] x_t + \left[-\beta + b\left(c_{h-1};\nu,\mu^*\right) + d_{h-1}\right]\right\}
\end{aligned}
$$

with $\rho^* = \dfrac{\rho}{(1-\Gamma_t\mu)^2}$ and $\mu^* = \dfrac{\mu}{1-\Gamma_t\mu}$. Thus, if we identify the coefficients, we find:

$$
\begin{cases}
c_h &= -\alpha + [a(c_{h-1} + \Gamma_t;\rho,\mu) - a(\Gamma_t;\rho,\mu)] \\
&= -\alpha + a(c_{h-1};\rho^*,\mu^*)\,, \\
d_h &= -\beta + [b(c_{h-1} + \Gamma_t;\nu,\mu) - b(\Gamma_t;\nu,\mu)] + d_{h-1} \\
&= -\beta + b(c_{h-1};\nu,\mu^*) + d_{h-1}\,.
\end{cases}
$$

From the condition $B(t,t) = 1$ we immediately find $c_0 = 0, d_0 = 0$.

ii) We have that the scalar latent factor x_t has an historical dynamics given by the following ARG(p) process:

$$
\frac{x_{t+1}}{\mu} \,|\, z_{t+1} \sim \gamma(\nu + z_{t+1}, 1)\,, \;\; \nu > 0\,,
$$

$$
z_{t+1} \,|\, \underline{x_t} \sim \mathcal{P}\left(\frac{\rho_1 x_t + \ldots + \rho_p x_{t-p+1}}{\mu}\right)\,, \;\; \rho_i = \beta_i\,\mu\,, \;\; i \in \{1,\ldots,p\}\,.
$$

that is, $x_{t+1} = \nu\mu + \rho' X_t + \varepsilon_{t+1}$ with $\rho = (\rho_1, \ldots, \rho_p)'$ and $X_t = (x_t, \ldots, x_{t-p+1})'$ (weak positive AR(p) representation). This means that, under $\mathbb{P}$, the Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$
\begin{aligned}
E\left[\exp(u x_{t+1}) \,|\, \underline{x_t}\right] &= \exp\left[\frac{u}{1-u\mu}\left(\rho_1 \, x_t + \ldots + \rho_p \, x_{t-p+1}\right) - \nu \log(1-u\mu)\right] , \\
&= \exp\left[\frac{u}{1-u\mu}\, \rho' \, X_t - \nu \log(1-u\mu)\right] , \\
&= \exp\left[a(u;\rho,\mu)' X_t + b(u;\nu,\mu)\right] ,
\end{aligned}
$$

and the Laplace transform of ε_{t+1}, conditionally to $\underline{\varepsilon_t}$, is given by:

$$
\begin{aligned}
E\left[\exp(u\varepsilon_{t+1}) \,|\, \underline{\varepsilon_t}\right] &= E\left\{\exp\left[u(x_{t+1} - \nu\mu - \rho' \, X_t) \,|\, \underline{x_t}\right]\right\} , \\
&= \exp\left[a(u;\rho,\mu)' \, X_t + b(u;\nu,\mu) - u\,(\nu\,\mu + \rho' \, X_t)\right] , \\
&= \exp\left[(a(u;\rho,\mu) - u\rho)' \, X_t + b(u;\nu,\mu) - u\,\nu\,\mu\right] .
\end{aligned}
$$

Given that the SDF is:

$$
M_{t,t+1} = \exp\left[-\beta - \alpha' X_t + \Gamma_t \, \varepsilon_{t+1}\right] \exp\left[-a\left(\Gamma_t;\rho,\mu\right)' \, X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho' \, X_t\right)\right] ,
$$

we can now write the following:

$$
\begin{aligned}
B(t,t+h) &= \exp(c_h' \, X_t + d_h) \\
&= E_t[M_{t,t+1} \cdots M_{t+H-1,t+H}] \\
&= E_t[M_{t,t+1} B(t+1,t+h)] \\
&= E_t\left\{\exp\left[-\beta - \alpha' X_t + \Gamma_t \, \varepsilon_{t+1}\right] \exp\left[-a\left(\Gamma_t;\rho,\mu\right)' \, X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho' \, X_t\right)\right]\right. \\
&\qquad\qquad \left.\exp(c_{h-1}' \, X_{t+1} + d_{h-1})\right\} \\
&= \exp\left[-\beta - \alpha' X_t - a\left(\Gamma_t;\rho,\mu\right)' \, X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho' \, X_t\right) + d_{h-1}\right] \\
&\qquad\qquad E_t[\exp(\Gamma_t \, \varepsilon_{t+1} + c_{1,h-1} \, x_{t+1} + c_{2,h-1}' \, \widetilde{X}_t)]
\end{aligned}
$$

where $c_{1,h-1}$ is the first component of the p-dimensional vector $c_{h-1} = (c_{1,h-1}, c_{2,h-1}')'$ and where

$\widetilde{X}_t = (x_t, \ldots, x_{t-p+2})'$. This means that:

$$
\begin{aligned}
B(t,t+h) &= \exp(c_h' \, X_t + d_h) \\
&= \exp\left[-\beta - \alpha' X_t - a\left(\Gamma_t; \rho, \mu\right)' \, X_t - b\left(\Gamma_t; \nu, \mu\right) + \Gamma_t\left(\nu\,\mu + \rho' \, X_t\right) + c_{1,h-1}\left(\nu\mu + \rho' \, X_t\right)\right. \\
&\qquad \left. + c_{2,h-1}' \, \widetilde{X}_t + d_{h-1}\right] \, E_t[\exp((\Gamma_t + c_{1,h-1})\,\varepsilon_{t+1})] \\
&= \exp\left\{-\beta - \alpha' X_t + \left[a\left(\Gamma_t + c_{1,h-1}; \rho, \mu\right) - a\left(\Gamma_t; \rho, \mu\right) + \bar{c}_{h-1}\right]' X_t \right. \\
&\qquad \left. + \left[b\left(\Gamma_t + c_{1,h-1}; \nu, \mu\right) - b\left(\Gamma_t; \nu, \mu\right)\right] + d_{h-1}\right\} \\
&= \exp\left\{\left[-\alpha + a\left(c_{1,h-1}; \rho^*, \mu^*\right) + \bar{c}_{h-1}\right]' X_t + \left[-\beta + b\left(c_{h-1}; \nu, \mu^*\right) + d_{h-1}\right]\right\}
\end{aligned}
$$

where $\bar{c}_{h-1} = (c_{2,h-1}', 0)'$ and with $\rho^* = \dfrac{\rho}{(1-\Gamma_t\mu)^2}$ and $\mu^* = \dfrac{\mu}{1-\Gamma_t\mu}$. Thus, if we identify the coefficients, we find:

$$
\left\{
\begin{aligned}
c_h &= -\alpha + [a(c_{1,h-1} + \Gamma_t; \rho, \mu) - a(\Gamma_t; \rho, \mu)] + \bar{c}_{h-1} \\
&= -\alpha + a(c_{1,h-1}; \rho^*, \mu^*) + \bar{c}_{h-1}\,, \\
d_h &= -\beta + [b(c_{1,h-1} + \Gamma_t; \nu, \mu) - b(\Gamma_t; \nu, \mu)] + d_{h-1} \\
&= -\beta + b(c_{1,h-1}; \nu, \mu^*) + d_{h-1}\,.
\end{aligned}
\right.
$$

From the condition $B(t,t) = 1$ we immediately find $c_0 = 0, d_0 = 0$.

Exercise N° 11 [The ARG(p) Risk-Neutral Laplace Transform].

Let us consider the scalar latent factor x_t which is described, under $\mathbb{P}$, by the following ARG(1) process:

$$
\frac{x_{t+1}}{\mu} \mid z_{t+1} \sim \gamma(\nu + z_{t+1}, 1)\,, \quad \nu > 0\,,
$$

$$
z_{t+1} \mid \underline{x_t} \sim \mathcal{P}\left(\frac{\rho x_t}{\mu}\right)\,, \quad \rho = \beta\,\mu\,.
$$

This means that its conditional historical Laplace transform is given by:

$$
\begin{aligned}
E\left[\exp(u x_{t+1}) \mid \underline{x_t}\right] &= \exp\left[\frac{\rho u}{1-u\mu}\, x_t - \nu \log(1-u\mu)\right]\,, \\
&= \exp\left[a(u;\rho,\mu)\, x_t + b(u;\nu,\mu)\right]\,.
\end{aligned}
$$

Let us assume, moreover, that the one-period SDF $M_{t,t+1}$ is given by:

$$
M_{t,t+1} = \exp\left[-\beta - \alpha x_t + \Gamma_t\,\varepsilon_{t+1}\right] \exp\left[-a\left(\Gamma_t;\rho,\mu\right)\, x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\, x_t\right)\right].
$$

The conditional Laplace transform, under the risk-neutral equivalent martingale measure $\mathbb{Q}$ (we use the money-market account as numéraire!) is given by:

$$E_t^{\mathbb{Q}}[\exp(ux_{t+1})] = E_t\left[\frac{M_{t,t+1}}{E_t(M_{t,t+1})}\exp(ux_{t+1})\right]$$

$$= E_t\left\{\exp\left[\Gamma_t\,\varepsilon_{t+1} - a\left(\Gamma_t;\rho,\mu\right)\,x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\,x_t\right) + ux_{t+1}\right]\right\}$$

$$= \exp\left[-a\left(\Gamma_t;\rho,\mu\right)\,x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\,x_t\right) + u\left(\nu\,\mu + \rho\,x_t\right)\right]\;E_t\left\{\exp\left[\left(\Gamma_t + u\right)\varepsilon_{t+1}\right]\right\}$$

$$= \exp\left\{\left[a\left(\Gamma_t + u;\rho,\mu\right) - a\left(\Gamma_t;\rho,\mu\right)\right]x_t + \left[b\left(\Gamma_t + u;\nu,\mu\right) - b\left(\Gamma_t;\nu,\mu\right)\right]\right\}$$

$$= \exp\left\{a\left(u;\rho^*,\mu^*\right)\,x_t + b\left(u;\nu,\mu^*\right)\right\}$$

where:

$$a(u+\Gamma_t;\rho,\mu) - a(\Gamma_t;\rho,\mu) = a(u;\rho^*,\mu^*)$$

$$b(u+\Gamma_t;\nu,\mu) - b(\Gamma_t;\nu,\mu) = b(u;\nu,\mu^*)$$

$$\text{with } \rho^* = \frac{\rho}{(1-\Gamma_t\mu)^2},\;\; \mu^* = \frac{\mu}{1-\Gamma_t\mu}\,.$$

Let us consider now a scalar latent factor x_t with an historical dynamics described by the following ARG(p) process:

$$\frac{x_{t+1}}{\mu}\,|\,z_{t+1} \sim \gamma(\nu + z_{t+1}, 1)\,,\;\; \nu > 0\,,$$

$$z_{t+1}\,|\,\underline{x_t} \sim \mathcal{P}\left(\frac{\rho_1 x_t + \ldots + \rho_p x_{t-p+1}}{\mu}\right)\,,\;\; \rho_i = \beta_i\,\mu\,,\;\; i \in \{1,\ldots,p\}\,.$$

Its conditional historical Laplace transform is therefore given by:

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\frac{u}{1-u\mu}\,\rho'\,X_t - \nu\log(1-u\mu)\right]\,,$$

$$= \exp\left[a(u;\rho,\mu)'\,X_t + b(u;\nu,\mu)\right]\,.$$

where $\rho = (\rho_1,\ldots,\rho_p)'$ and $X_t = (x_t,\ldots,x_{t-p+1})'$. Let us assume, moreover, that the one-period SDF $M_{t,t+1}$ is given by:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' X_t + \Gamma_t\,\varepsilon_{t+1}\right]\,\exp\left[-a\left(\Gamma_t;\rho,\mu\right)'\,X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho'\,X_t\right)\right]\,,$$

where $\Gamma_t = \gamma_o + \gamma' X_t$. The conditional Laplace transform, under the risk-neutral equivalent martingale measure $\mathbb{Q}$ is given by:

$$E_t^{\mathbb{Q}}[\exp(ux_{t+1})] = E_t\left[\frac{M_{t,t+1}}{E_t(M_{t,t+1})}\exp(ux_{t+1})\right]$$

$$= \quad E_t\left\{\exp\left[\Gamma_t\,\varepsilon_{t+1} - a\left(\Gamma_t;\rho,\mu\right)'\,X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho'\,X_t\right) + ux_{t+1}\right]\right\}$$

$$= \quad \exp\left[-a\left(\Gamma_t;\rho,\mu\right)'\,X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho'\,X_t\right) + u\left(\nu\,\mu + \rho'\,X_t\right)\right]\,E_t\left\{\exp\left[\left(\Gamma_t + u\right)\varepsilon_{t+1}\right]\right\}$$

$$= \quad \exp\left\{\left[a\left(\Gamma_t + u;\rho,\mu\right) - a\left(\Gamma_t;\rho,\mu\right)\right]'\,X_t + \left[b\left(\Gamma_t + u;\nu,\mu\right) - b\left(\Gamma_t;\nu,\mu\right)\right]\right\}$$

$$= \quad \exp\left\{a\left(u;\rho^*,\mu^*\right)'\,X_t + b\left(u;\nu,\mu^*\right)\right\}$$

where:

$$a(u+\Gamma_t;\rho,\mu) - a(\Gamma_t;\rho,\mu) = a(u;\rho^*,\mu^*)$$

$$b(u+\Gamma_t;\nu,\mu) - b(\Gamma_t;\nu,\mu) = b(u;\nu,\mu^*)$$

$$\text{with } \rho^* = \frac{1}{(1-\Gamma_t\mu)^2}\,\rho\,,\ \ \mu^* = \frac{\mu}{1-\Gamma_t\mu}\,.$$