Fixed Income and Credit Risk : exercise sheet n° 05

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Exercise N° 01 [Exponential-affine ZCB Pricing Formula].

Let us consider a discrete-time Gaussian term structure model, in which the K-dimensional factor x_{t+1} has an historical dynamics described by the Gaussian VAR(p) process:

$$\begin{aligned} x_{t+1} &= \nu + \Phi_1 x_t + \ldots + \Phi_p x_{t+1-p} + \Sigma\varepsilon_{t+1} \\ &= \nu + \Phi X_t + \Sigma\varepsilon_{t+1} , \end{aligned}$$

where ε_{t+1} is a Gaussian white noise with $\mathcal{N}(0, I_K)$ distribution. We have that ν is a K-dimensional vector, $\Phi = [\Phi_1, \ldots, \Phi_p]$ is a (K, Kp)-dimensional matrix, $X_t = [x'_t, \ldots, x'_{t+1-p}]'$ is a (Kp)-dimensional vector. We also have that Σ is a (K, K) lower triangular matrix : it is the Choleski decomposition of $V_t[x_{t+1}] = \Omega$.

Let us also assume that the stochastic discount factor (SDF) $M_{t,t+1}$ for the period $(t, t+1)$ has the following exponential-affine specification:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' X_t + \Gamma'_t\, \varepsilon_{t+1} - \frac{1}{2}\Gamma'_t\Gamma_t\right] ,$$

where $\Gamma_t = \gamma_o + \widetilde{\Gamma} X_t$, $\Gamma_t = [\,\Gamma_{1,t}, \ldots, \Gamma_{K,t}]'$. Prove that the price at date t of the zero-coupon bond with time to maturity h is :

$$B(t,h) = \exp(C'_h X_t + D_h) , \quad h \geq 1 ,$$

where C_h and D_h satisfies the recursive equations :

$$\begin{aligned} C_h &= -\alpha + \widetilde{\Phi}' C_{h-1} + (\Sigma\widetilde{\Gamma})' C_{1,h-1} \\ &= -\alpha + \widetilde{\Phi}^{*\prime} c_{h-1} , \\ D_h &= -\beta + C'_{1,h-1}(\nu + \Sigma\gamma_o) + \tfrac{1}{2} C'_{1,h-1}(\Sigma\Sigma')C_{1,h-1} + D_{h-1} , \end{aligned}$$

with :

$$\widetilde{\Phi}^* = \begin{bmatrix} \Phi_1 + \Sigma\gamma_1 & \ldots & \ldots & \Phi_{p-1} + \Sigma\gamma_{p-1} & \Phi_p + \Sigma\gamma_p \\ I_K & \mathbf{0}_K & \ldots & \mathbf{0}_K & \mathbf{0}_K \\ \mathbf{0}_K & I_K & \ldots & \mathbf{0}_K & \mathbf{0}_K \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0}_K & \ldots & \ldots & I_K & \mathbf{0}_K \end{bmatrix} \text{ is a } (Kp, Kp) \text{ matrix} , \qquad (1)$$

and where the initial conditions are $C_0 = 0, D_0 = 0$ (or $C_1 = -\alpha, D_1 = -\beta$), where $C_{1,h}$ indicates the vector of the first K components of the (Kp)-dimensional vector C_h.

Exercise N° 02 [A different derivation of the Gaussian ATSM - Scalar case].

During Lecture 4 we have seen how to calculate the no-arbitrage yield-to-maturity formula $R(t,h)$ for AR(p) and VAR(p) Factor-Based Term Structure Models characterized by an exponential-affine SDF $M_{t,t+1}$. Let us consider, for ease of exposition, a Gaussian AR(1) setting. The strategy we have presented was the following :

a) make an assumption about the historical dynamics of the factor (x_t) representing the information the investor uses to price ZCB at any date t and for any residual maturity h. We have assumed that, under the historical probability $\mathbb{P}$, the factor (x_t) is described by a Gaussian AR(p) or VAR(p) process. In the Gaussian AR(1) case, this means that:

$$x_{t+1} = \nu + \varphi x_t + \sigma \varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim \mathcal{N}(0,1) \;\; (\text{under } \mathbb{P})\,.$$

b) Make an assumption about the functional form and the parametric specification of the one-period SDF $M_{t,t+1}$. In the Gaussian AR(1) case, we have assumed:

$$\begin{aligned} M_{t,t+1} &= \exp\left[-\beta - \alpha x_t + \Gamma_t \varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t^2\right]\,, \;\; (\text{SDF}) \\ \Gamma_t &= \Gamma(x_t) = (\gamma_o + \gamma x_t)\,, \end{aligned}$$

and under the absence of arbitrage opportunities principle, we have found $r_t = \beta + \alpha x_t$ and, more generally:

$$\begin{aligned} R(t,h) &= -\frac{c_h}{h} x_t - \frac{d_h}{h}\,, \\ c_h &= -\alpha + \varphi c_{h-1} + c_{h-1}\sigma\gamma = -\alpha + (\varphi + \sigma\gamma) c_{h-1}\,, \\ d_h &= -\beta + c_{h-1}(\nu + \gamma_o \sigma) + \tfrac{1}{2} c_{h-1}^2 \sigma^2 + d_{h-1}\,, \\ & c_0 = 0, d_0 = 0\,. \end{aligned}$$

The purpose of this exercise is to show that we only need to make a proper assumption about the dynamics of the latent factor x_t and (if they are not the same) of the short rate r_t under the risk-neutral probability measure $\mathbb{Q}$ in order to find the same yield-to-maturity formula $R(t,h)$. Prove this sentence in the following two cases: *i*) x_t is, under $\mathbb{Q}$, a latent factor following a Gaussian AR(1) process, and $r_t = \beta + \alpha x_t$; *ii*) $x_t = r_t$ is, under $\mathbb{Q}$, an observable factor following a Gaussian AR(1) process.

Exercise N° 03 [SDF, historical and risk-neutral dynamics and change of measure].

We consider an economy between dates 0 and T. The new information in the economy at date t is denoted by w_t and the overall information at date t is $\underline{w}_t = (w_t, w_{t-1}, ..., w_0)$. The random variable w_t is called a factor or a state vector, and it may be observable, partially observable or unobservable by the econometrician. The size of w_t is K. During Lecture 4, w_t has been specified as a Gaussian VAR(p) process with $K \geq 1$ and $p \geq 1$.

The historical dynamics of w_t is defined by the joint distribution of $\underline{w}_T$, denoted by $\mathbb{P}$, or by the conditional p.d.f. (with respect to some measure):

$$f_t(w_{t+1}|\underline{w}_t)\,,$$

or by the conditional Laplace transform (L.T.):

$$\varphi_t(u|\underline{w}_t) = E[\exp(u'w_{t+1})\,|\,\underline{w}_t]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^K$ (containing zero). We also introduce the conditional Log-Laplace transform :

$$\psi_t(u|\underline{w}_t) = \text{Log}[\varphi_t(u|\underline{w}_t)]\,.$$

The conditional expectation operator, given $\underline{w}_t$, is denoted by E_t. $\varphi_t(u|\underline{w}_t)$ and $\psi_t(u|\underline{w}_t)$ will be also denoted by $\varphi_t(u)$ and $\psi_t(u)$.

(i) Specify the one-period SDF $M_{t,t+1}(\underline{w}_{t+1})$ in such a way that $E_t[M_{t,t+1}] = \exp(-r_t)$, where r_t is the short rate between t and $t+1$ (known in t).

(ii) Specify the one-period change of probability measures $\dfrac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}}$ and $\dfrac{d\mathbb{P}_{t,t+1}}{d\mathbb{Q}_{t,t+1}}$ as a function of historical and risk-neutral factor's dynamics.

Exercise N° 04 [Exercise N° 03, continued].

Let us consider the Gaussian AR(1) latent process x_t, under the probability measure $\mathbb{Q}$, introduced in Exercise N° 02.

(i) On the basis of the results presented in Exercise N° 03, specify the SDF $M_{t,t+1} = M_{t,t+1}(\eta_{t+1})$, where $\eta_t \sim \mathcal{N}(0,1)$ under $\mathbb{Q}$.

(ii) Determine the historical dynamics of (x_t). For which reason, in your opinion, is important to know the historical dynamics of (x_t) given that we can determine the ZCB pricing formula directly working under $\mathbb{Q}$?

Exercise N° 05 [No-arbitrage restrictions for the short and long rate].

Let us assume to have a bivariate Gaussian VAR(1) Factor-Based term structure models, and let us assume that the factor x_t be given by $x_t = (r_t, R_t)'$ where $r_t = R(t, t+1)$ is the yield with the shortest maturity in our data base (it is the short rate) and where R_t is the long rate $R(t, t+H)$ (i.e., the yield with the longest maturity in our data base). This Gaussian VAR(1) ATSM can be summarized as follows:

$$
\begin{aligned}
x_{t+1} &= \nu + \Phi x_t + \Sigma\varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim \mathcal{N}(0, I_2) \;\; \text{(under } \mathbb{P}\text{)} \\
M_{t,t+1} &= \exp\left[-\beta - \alpha' x_t + \Gamma_t'\varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t'\Gamma_t\right]\,, \;\; \text{(SDF)} \\
\Gamma_t &= \Gamma(x_t) = (\gamma_o + \gamma\, x_t)\,, \\
R(t, t+h) &= -\frac{C_h{}'}{h}x_t - \frac{D_h}{h}\,, \\
C_h &= -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} = -\alpha + \Phi^{*'} C_{h-1}\,, \\
D_h &= -\beta + C_{h-1}'(\nu + \Sigma\gamma_o) + \tfrac{1}{2}C_{h-1}'(\Sigma\Sigma')C_{h-1} + D_{h-1}\,, \\
& C_0 = 0, D_0 = 0\,.
\end{aligned}
$$

Write the complete set of no-arbitrage restrictions that this model has to satisfy.

Exercise N° 06 [Conditional distribution of yields when the factor is Gaussian AR(p)].

Let us consider the following Gaussian AR(p) Factor Based Term structure model:

$$
\begin{aligned}
x_{t+1} &= \nu + \varphi_1 x_t + \ldots + \varphi_p x_{t+1-p} + \sigma\varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim \mathcal{N}(0, 1) \;\; \text{(under } \mathbb{P}\text{)} \\
M_{t,t+1} &= \exp\left[-\beta - \alpha' X_t + \Gamma_t\varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t^2\right]\,, \;\; \text{(SDF)}\,, \;\; X_t = [\,x_t, \ldots, x_{t+1-p}\,]' \\
r_t &= \beta + \alpha' X_t\,, \;\; \Gamma_t = \Gamma(x_t) = (\gamma_o + \gamma' X_t)\,, \\
R(t, h) &= -\frac{c_h{}'}{h}X_t - \frac{d_h}{h}\,, \\
c_h &= -\alpha + \Phi' c_{h-1} + c_{1,h-1}\sigma\gamma = -\alpha + \Phi^{*'} c_{h-1}\,, \\
d_h &= -\beta + c_{1,h-1}(\nu + \gamma_o\sigma) + \tfrac{1}{2}c_{1,h-1}^2\sigma^2 + d_{h-1}\,, \\
& c_0 = 0, d_0 = 0\,,
\end{aligned}
$$

where (x_t) is latent, $\gamma = (\gamma_1, \ldots, \gamma_p)'$ and where:

$$\Phi^* = \begin{bmatrix} \varphi_1 + \sigma\gamma_1 & \ldots & \ldots & \varphi_{p-1} + \sigma\gamma_{p-1} & \varphi_p + \sigma\gamma_p \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & 1 & 0 \end{bmatrix}.$$

For a given residual maturity h, which is the historical dynamics of the yield process $R = [\,R(t,h),\ 0 \leq t < T\,]$?

Exercise N° 07 [Conditional p.d.f. of yields when the factor is Gaussian VAR(1)].

When we have presented Gaussian VAR(1) Factor-Based Term Structure models, we have determined the link between the ZCB price $B(t,h)$ (or the yield $R(t,h)$) and the factor (x_t) at a given point in time t and for any residual maturity h. The purpose of this exercise is to use the historical conditional p.d.f. of the K-dimensional VAR(1) factor (x_t) (see exercise N° 04 for the notation) to determine the historical conditional p.d.f. of any yield over time (i.e., t is varying and h is fixed). We also briefly review the method of Pearson and Sun (1994) that use the yield dynamics to obtain a Maximum Likelihood estimation of our Gaussian VAR(1) ATSM.

Exercise N° 08 [Working with the Non-centered Gamma Distribution].

Let us assume that Y is a Non-centered Gamma random variable with parameters $\nu > 0$, $\beta > 0$ and $\mu > 0$, i.e. $Y \sim \widetilde{\gamma}(\nu, \beta, \mu)$. We know that this is equivalent to the existence of a random variable $Z \sim \mathcal{P}(\beta)$ such that:

$$\begin{cases} \dfrac{Y}{\mu} \,|\, Z \sim \gamma(\nu + Z, 1)\,, \;\; \nu > 0\,, \\ Z \,\sim\, \mathcal{P}(\beta)\,, \;\; \beta > 0\,, \mu > 0\,, \end{cases} \iff \begin{cases} Y \,|\, Z \sim \gamma(\nu + Z, \mu)\,, \;\; \nu > 0\,, \\ Z \,\sim\, \mathcal{P}(\beta)\,, \;\; \beta > 0\,, \mu > 0\,, \end{cases}$$

where β is the non-centrality parameter. Prove that:

- $E[Y] = \nu\,\mu + \beta\,\mu$ and $V[Y] = \nu\,\mu^2 + 2\mu^2\beta$ (**mean** and **variance**);
- $E[\exp(uY)] = \exp\left[-\nu \log(1 - u\,\mu) + \beta \dfrac{u\,\mu}{1 - u\,\mu}\right]$, for $u < 1/\mu$ (**Laplace transform**);

Exercise N° 09 [Working with the ARG(1) Process].

Let us consider an ARG(1) process (x_t) defined as:

$$\frac{x_{t+1}}{\mu} | z_{t+1} \sim \gamma(\nu + z_{t+1}, 1)\,, \;\; \nu > 0\,,$$

$$z_{t+1} | x_t \sim \mathcal{P}(\rho x_t / \mu)\,, \;\; \rho > 0\,, \mu > 0\,, \rho = \beta\,\mu$$

Show that:

- $E(x_{t+1}\,|\,x_t) = \nu\mu + \rho x_t$ and $V(x_{t+1}\,|\,x_t) = \nu\mu^2 + 2\mu\rho x_t$.
- $E(x_t) = \dfrac{\nu\mu}{1-\rho}$ and $V(x_t) = \dfrac{\nu\mu^2}{(1-\rho)^2}$.
- $E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\dfrac{\rho u}{1-u\mu}\,x_t - \nu\log(1-u\mu)\right]$.
- $E(\varepsilon_{t+1}\,|\,\varepsilon_t) = 0$, $V(\varepsilon_{t+1}\,|\,\varepsilon_t) = \nu\mu^2 + 2\mu\rho x_t$, $E(\varepsilon_t) = 0$ and $V(\varepsilon_t) = \nu\mu^2 + 2\nu\mu^2\dfrac{\rho}{1-\rho}$.

Exercise N° 10 [The ARG(1) and ARG(p) Positive Affine Yield Curves].

Let us consider a scalar latent factor x_{t+1} whose historical dynamics is described by the following ARG(p) process, with $p \geq 1$:

$$\frac{x_{t+1}}{\mu}\,|\,z_{t+1} \sim \gamma(\nu + z_{t+1}, 1)\,, \quad \nu > 0\,,$$

$$z_{t+1}\,|\,\underline{x_t} \sim \mathcal{P}\left(\frac{\rho_1 x_t + \ldots + \rho_p x_{t-p+1}}{\mu}\right)\,, \quad \rho_i = \beta_i\,\mu\,, \quad i \in \{1,\ldots,p\}\,.$$

i) Show that, when $p = 1$, and the one-period stochastic discount factor (SDF) $M_{t,t+1}$ is given by:

$$M_{t,t+1} = \exp\left[-\beta - \alpha x_t + \Gamma_t\,\varepsilon_{t+1}\right]\,\exp\left[-a\left(\Gamma_t;\rho,\mu\right)\,x_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho\,x_t\right)\right],$$

with $\Gamma_t = \gamma_o + \gamma\,x_t$, then the positive affine term structure of interest rates is given by:

$$R(t,t+h) = -\frac{1}{h}\log B(t,t+h) = -\frac{c_h}{h}x_t - \frac{d_h}{h}\,, \quad h \geq 1\,,$$

with:

$$\begin{cases} c_h &= -\alpha + \left[a(c_{h-1}+\Gamma_t;\rho,\mu) - a(\Gamma_t;\rho,\mu)\right] \\ &= -\alpha + a(c_{h-1};\rho^*,\mu^*)\,, \\ d_h &= -\beta + \left[b(c_{h-1}+\Gamma_t;\nu,\mu) - b(\Gamma_t;\nu,\mu)\right] + d_{h-1} \\ &= -\beta + b(c_{h-1};\nu,\mu^*) + d_{h-1}\,, \end{cases}$$

and with initial conditions $c_0 = 0, d_0 = 0$ (or $c_1 = -\alpha, d_1 = -\beta$).

ii) Now, let us assume that $p > 1$ and that the SFD is given by:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' X_t + \Gamma_t\,\varepsilon_{t+1}\right]\,\exp\left[-a\left(\Gamma_t;\rho,\mu\right)'\,X_t - b\left(\Gamma_t;\nu,\mu\right) + \Gamma_t\left(\nu\,\mu + \rho'\,X_t\right)\right],$$

with $\Gamma_t = \gamma_o + \gamma' x_t$. Show that the positive affine term structure of interest rates is given by:

$$R(t,t+h) = -\frac{1}{h}\log B(t,t+h) = -\frac{{c_h}'}{h}X_t - \frac{d_h}{h}\,, \quad h \geq 1\,,$$

with:

$$\begin{cases} c_h & = & -\alpha + [a(c_{1,h-1} + \Gamma_t; \rho, \mu) - a(\Gamma_t; \rho, \mu)] + \bar{c}_{h-1} \\ & = & -\alpha + a(c_{1,h-1}; \rho^*, \mu^*) + \bar{c}_{h-1}\,, \\ d_h & = & -\beta + [b(c_{1,h-1} + \Gamma_t; \nu, \mu) - b(\Gamma_t; \nu, \mu)] + d_{h-1} \\ & = & -\beta + b(c_{1,h-1}; \nu, \mu^*) + d_{h-1}\,, \end{cases}$$

and where $\bar{c}_{h-1} = (c_{2,h-1}, \ldots, c_{p,h-1}, 0)'$, and with initial conditions $c_0 = 0, d_0 = 0$ (or $c_1 = -\alpha, d_1 = -\beta$).

Exercise N$^\circ$ 11 [The ARG(p) Risk-Neutral Laplace Transform].

Determine Risk-Neutral Laplace transform of the ARG(p) process, presented in the previous exercise, for $p = 1$ and $p > 1$.