Fixed Income and Credit Risk : solutions for exercise sheet n° 04

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Exercise N° 01 [Exponential-affine ZCB Pricing Formula].

Given that $M_{t,t+1}$ is exponential-affine in ε_{t+1} (i.e. x_{t+1}) and that the conditional Laplace transform of x_{t+1} is exponential-affine in the conditioning variable (x_t) we suggest that the ZCB pricing formula at date t be an exponential-affine function of x_t and then "we check if it works". We proceed in the following way:

a) We suggest $B(t,t+h) = \exp(c_h' \, X_t + d_h)$ and we (equivalently) rewrite the pricing formula in terms of the payoff $B(t+1,t+h) = \exp(c_{h-1}' \, X_{t+1} + d_{h-1})$:

$$\begin{aligned} B(t,t+h) &= \exp(c_h' \, X_t + d_h) \\ &= E_t[M_{t,t+1} \cdots M_{t+h-1,t+h}] \\ &= E_t[M_{t,t+1} B(t+1,t+h)] \\ &= E_t\left[\exp\left(-\beta - \alpha' \, X_t + \Gamma_t \, \varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma_t^2\right) \exp(c_{h-1}' \, X_{t+1} + d_{h-1})\right] , \end{aligned}$$

b) we do the algebra (calculating the conditional Laplace transform) obtaining:

$$\begin{aligned} & B(t,t+h) \\ = \; & \exp(c_h{}' X_t + d_h) \\ = \; & \exp\left[-\beta - \alpha' X_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1}\right] \times E_t[\exp\left(\Gamma_t \varepsilon_{t+1} + c_{h-1}' X_{t+1}\right)] \\ = \; & \exp\left[-\beta - \alpha' X_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1} + c_{h-1}'(\Phi X_t + \tilde{\nu})\right] \times \; E_t[\exp\left(\Gamma_t + \sigma c_{1,h-1}\right)\varepsilon_{t+1})] \\ = \; & \exp\left[(-\alpha + \Phi' c_{h-1} + c_{1,h-1}\sigma\gamma)' X_t \right. \\ & \quad \left. + \left(-\beta + c_{1,h-1}\nu + \tfrac{1}{2} c_{1,h-1}^2 \sigma^2 + \gamma_o c_{1,h-1}\sigma + d_{h-1}\right)\right] , \end{aligned}$$

c) and by identifying the coefficients we find the recursive relations for c_h and d_h characterizing the pricing formula $B(t,t+h) = \exp(c_h' \, X_t + d_h)$.

Now, the last elements we need to completely determine the pricing formula are the starting conditions for c_h and d_h. We proceed as follows:

given that, by definition of ZCB, we have $B(t,t) = 1$, then

$$\exp(c_0' X_t + d_0) = 1 \iff (c_0' X_t + d_0) = 0 \;\; \forall \;\; X_t \iff c_0 = 0\,, \;\; d_0 = 0\,.$$

We can also equivalently write:

given that, by definition of ZCB, we have $B(t,t+1) = \exp(-\,r_t)$, then

$$\exp(c_1' X_t + d_1) = \exp(-\,r_t) \iff (c_1' X_t + d_1) = -r_t \;\; \forall \;\; X_t \iff c_1 = -\,\alpha\,, \;\; d_1 = -\,\beta\,.$$

Exercise N° 02 [Identification Issue in latent factor Gaussian ATSMs].

We have the following family of Gaussian AR(p) Factor-Based term structure models:

$$\begin{aligned}
x_{t+1} &= \nu + \varphi x_t + \sigma\varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim \mathcal{N}(0,1) \;\; \text{(under } \mathbb{P}\text{)} \\
M_{t,t+1} &= \exp\left[-\beta - \alpha x_t + \Gamma_t \varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t^2\right]\,, \;\; \text{(SDF)} \\
r_t &= \beta + \alpha x_t\,, \;\; \Gamma_t = \Gamma(x_t) = (\gamma_o + \gamma x_t)\,, \\
R(t,h) &= -\frac{c_h}{h} x_t - \frac{d_h}{h}\,, \\
c_h &= -\alpha + \varphi c_{h-1} + c_{h-1}\sigma\gamma = -\alpha + (\varphi + \sigma\gamma)c_{h-1}\,, \\
d_h &= -\beta + c_{h-1}(\nu + \gamma_o\sigma) + \tfrac{1}{2}c_{h-1}^2\sigma^2 + d_{h-1}\,, \\
c_0 &= 0, d_0 = 0\,.
\end{aligned}$$

We have seen that the yield-to-maturity formula $R(t,h)$ is completely determined by the specification of the historical dynamics of x_{t+1} (Gaussian AR(1)) and by the specification of the one-period SDF $M_{t,t+1}$ (exponential-affine in ε_{t+1}). The identification issue, associated to the specification of yield-to-maturity formula, is therefore given by the fact that different (infinitely many) set of parameter values (ν, φ, σ) and $(\beta, \alpha, \gamma_o, \gamma)$ can generate the same theoretical $R(t,h)$. This means that, from a given time series of observations $R^o(t,h)$, we cannot detect the unique set of parameters such that the distance between $R(t,h)$ and $R^o(t,h)$ is minimized, given that several different set of parameters determine the same $R(t,h)$.

More formally : for arbitrary real constants μ_1 and μ_2, if we replace:

a) x_t by $\bar{x}_t = \mu_1 + \mu_2 x_t$ (any Gaussian stochastic process can be represented as an affine transformation of the centered and normalized one),

b.1) γ by $\dfrac{\gamma}{\mu_2}$,

$b.2)$ γ_o by $\gamma_o - \dfrac{\mu_1}{\mu_2}\gamma' e$,

$b.3)$ α by $\dfrac{\alpha}{\mu_2}$

$b.4)$ and β by $\beta - \dfrac{\mu_1}{\mu_2}\alpha' e$.

we obtained the same SDF $M_{t,t+1}$ dynamics (depending on the factor dynamics) and therefore we generate the same yield $R(t,h)$ (remember that $B(t,h) = E_t[M_{t,t+1} \dots M_{t+h-1,t+h}]$). In other words, for a starting parametric specification of x_t and $M_{t,t+1} = M_{t,t+1}(x_t)$, we have that, after the parametric transformations $a)$, $b.1) - b.4)$, we obtain a new latent factor $\bar{x}_t$ and SDF $\bar{M}_{t,t+1}(\bar{x}_t)$ such that $M_{t,t+1}(x_t) = M_{t,t+1}(\bar{x}_t)$ for any t. Thus, $E_t[M_{t,t+1} \dots M_{t+h-1,t+h}] = E_t[\bar{M}_{t,t+1} \dots \bar{M}_{t+h-1,t+h}]$ for any t and h and the theoretical yields are therefore the same.
If x_t is not directly observed, we can assume for instance, as far as the term structure is concerned, that $\nu = 0$ and $\sigma = 1$, or $\beta = 0$ and $\alpha = 1$. In this way the identification problem is solved. This result is easily generalized to the case of a Gaussian AR(p) process [see Monfort and Pegoraro (2007), Section 2.4].

Exercise N° 03 [Excess Returns of Zero-Coupon Bonds].

Given that $B(t,T) = \exp(c'_{T-t} X_t + d_{T-t})$, we can write:

$$\begin{aligned}
\rho(t+1,\, T) &= \log\left[B(t+1,\, T)\right] - \log\left[B(t,\, T)\right] \\
&= c'_{T-t-1} X_{t+1} + d_{T-t-1} - c'_{T-t} X_t - d_{T-t} \\
&= c'_{T-t-1}\left[X_{t+1} - \Phi X_t - \tilde{\nu}\right] + (\beta + \alpha' X_t) - \,\sigma c_{1,T-t-1}(\gamma_o + \gamma' X_t) - \tfrac{1}{2} c^2_{1,T-t-1}\sigma^2 \\
&= (c_{1,T-t-1}\sigma)\varepsilon_{t+1} + (\beta + \alpha' X_t) - \sigma c_{1,T-t-1}(\gamma_o + \gamma' X_t) - \, \tfrac{1}{2} c^2_{1,T-t-1}\sigma^2\,.
\end{aligned}$$

Now, we have that, under the absence of arbitrage $r_t = (\beta + \alpha' X_t)$ and, consequently, the result is proved.

Exercise N° 04 [Risk-Neutral Laplace Transform of the Gaussian AR(p) Factor].

The risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$\begin{aligned}
E^{\mathbb{Q}}_t[\exp(u x_{t+1})] &= E_t\left[\tfrac{M_{t,t+1}}{E_t(M_{t,t+1})} \exp(u x_{t+1})\right] \\
&= E_t\left[\exp\left((\gamma_o + \gamma' X_t)\,\varepsilon_{t+1} - \tfrac{1}{2}\,(\gamma_o + \gamma' X_t)^2 + u x_{t+1}\right)\right] \\
&= \exp\left[u(\nu + \varphi' X_t) - \tfrac{1}{2}(\gamma_o + \gamma' X_t)^2\right] \\
&\qquad E_t[\exp((\gamma_o + \gamma' X_t + u\sigma)\varepsilon_{t+1})] \\
&= \exp\left[u[(\nu + \sigma\gamma_o) + (\varphi + \sigma\gamma)' X_t] + \tfrac{1}{2} u^2 \sigma^2\right]\ ,
\end{aligned}$$

where $\varphi = [\,\varphi_1, \ldots, \varphi_p]'$.

Exercise N° 05 [Risk-Neutral Zero-Coupon Bond Return Process].

We have that

$$\begin{aligned}
\rho(t+1,\, T) &= \log\left[B(t+1,\, T)\right] - \log\left[B(t,\, T)\right] \\
&= c'_{T-t-1}X_{t+1} + d_{T-t-1} - c'_{T-t}X_t - d_{T-t} \\
&= c'_{T-t-1}X_{t+1} - (-\alpha' + c'_{T-t-1}\Phi^*)X_t + \beta - c_{1,T-t-1}\nu^* - \tfrac{1}{2}c^2_{1,T-t-1}\sigma^2 \\
&= c'_{T-t-1}\left[X_{t+1} - \Phi^* X_t - \tilde{\nu}^*\right] + r_t - \tfrac{1}{2}c^2_{1,T-t-1}\sigma^2 \\
&= r_t - \tfrac{1}{2}\omega(t+1,T)^2 - \omega(t+1,T)\eta_{t+1}
\end{aligned}$$

and the first part of the result is proved. Now, if we calculate $E_t^{\mathbb{Q}} \exp\left[\rho(t+1,\, T)\right]$ we have:

$$\begin{aligned}
E_t^{\mathbb{Q}} \exp\left[\rho(t+1,\, T)\right] &= E_t^{\mathbb{Q}} \exp\left[r_t - \tfrac{1}{2}\omega(t+1,T)^2 - \omega(t+1,T)\eta_{t+1}\right] \\
&= \exp[r_t - \tfrac{1}{2}\omega(t+1,T)^2]\, E_t^{\mathbb{Q}} \exp[-\omega(t+1,T)\eta_{t+1}] = \exp(r_t)
\end{aligned}$$

and therefore $\lambda_t^{\mathbb{Q}}(\rho, 1) = \log E_t^{\mathbb{Q}} \exp\left[\rho(t+1,\, T)\right] - r_t = 0$.

Exercise N° 06 [Yield Curve Shapes, Risk-Neutral Stationarity and Long Rates].

The different shapes that the yield curve formula $R(t, t+h) = -\dfrac{1}{h}[c'_h X_t + d_h]$ is able to reproduce depend crucially on the system of difference equations (c_h, d_h):

$$\left\{\begin{aligned}
c_h &= \Phi^{*'} c_{h-1} - \alpha \\
d_h &= -\beta + c_{1,h-1}\nu^* + \tfrac{1}{2}c^2_{1,h-1}\sigma^2 + d_{h-1}\,,
\end{aligned}\right.$$

with initial conditions $c_0 = 0$ and $d_0 = 0$. Let us consider in that exercise the case where x_{t+1} follows a Gaussian AR(1) and AR(2) process respectively.

$p = 1$ Let us consider that $x_t = r_t$ follows a Gaussian AR(1) process. In this case c_h satisfies the fist-order difference equation:

$$c_h = -1 + (\varphi + \sigma\gamma)c_{h-1}\,,$$

where $\sigma > 0$, γ and $|\varphi| < 1$ are scalar coefficients, and with a general solution, denoted $c(h)$, given by:

$$c(h) = -\left[\frac{1}{1 - (\varphi + \sigma\gamma)}\right]\left[1 - (\varphi + \sigma\gamma)^h\right] = -\left[\frac{1 - \varphi^{*h}}{1 - \varphi^*}\right],$$

which tends, for h increasing to infinity, to the limit:

$$\bar{c} = -\left[\frac{1}{1-\varphi^*}\right] ,$$

under the condition $|\varphi^*| < 1$, where $\varphi^* = (\varphi + \sigma\gamma)$ is the unique eigenvalue of the (scalar) matrix $\Phi^{*'}$. This means that this stability condition of the difference equation c_h coincide with the stationarity condition of the AR(1) process x_{t+1} under the risk-neutral probability $\mathbb{Q}$.

Now, observe that this condition implies $c(h) < 0$ for every $h > 0$. In addition, if $0 < \varphi + \sigma\gamma < 1$ (respectively, $-1 < \varphi + \sigma\gamma < 0$), the function $c(h)$ converges in decreasing (respectively, oscillating) towards $\bar{c}$.

With regard to d_h, it easy to verify that :

$$\begin{aligned} d(h) &= -\left[\frac{\nu^*}{1-\varphi^*}\right](h-1) + \left[\frac{\varphi^* - \varphi^{*h}}{1-\varphi^*}\right]\left[\frac{\nu^*}{1-\varphi^*} - \frac{\sigma^2}{(1-\varphi^*)^2}\right] \\ &\quad + \frac{\sigma^2}{2(1-\varphi^*)^2}\left[(h-1) + \frac{\varphi^{*2} - \varphi^{*2h}}{1-\varphi^{*2}}\right] . \end{aligned}$$

Consequently, the yield to maturity formula, for $p = 1$, is given by :

$$\begin{aligned} R(t,t+h) &= \frac{1}{h}\left[\frac{1-\varphi^{*h}}{1-\varphi^*}\right] r_t + \frac{(h-1)}{h}\left[\frac{\nu^*}{1-\varphi^*}\right] - \frac{1}{h}\left[\frac{\varphi^* - \varphi^{*h}}{1-\varphi^*}\right]\left[\frac{\nu^*}{1-\varphi^*} - \frac{\sigma^2}{(1-\varphi^*)^2}\right] \\ &\quad - \frac{\sigma^2}{2h(1-\varphi^*)^2}\left[(h-1) + \frac{\varphi^{*2} - \varphi^{*2h}}{1-\varphi^{*2}}\right] . \end{aligned}$$

$p = 2$ If the factor $x_t = r_t$ is a Gaussian AR(2) process, the recursive equation for c_h is described by a first-order (2×2) system of difference equations of the following type:

$$\begin{bmatrix} c_{1,h} \\ c_{2,h} \end{bmatrix} - \begin{bmatrix} \varphi_1 + \sigma\gamma_1 & 1 \\ \varphi_2 + \sigma\gamma_2 & 0 \end{bmatrix} \begin{bmatrix} c_{1,h-1} \\ c_{2,h-1} \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \tag{1}$$

substituting the first equation into the second, we find for $c_{1,h+1}$ the following second-order linear difference equation:

$$c_{1,h+1} = -1 + \varphi_1^* c_{1,h} + \varphi_2^* c_{1,h-1} , \tag{2}$$

where $\varphi_1^* = (\varphi_1 + \sigma\gamma_1)$ and $\varphi_2^* = (\varphi_1 + \sigma\gamma_2)$; under the condition that the two eigenvalues (λ_1, λ_2) of $\Phi^{*'}$ (or the inverse of the roots of $1 - \varphi_1^* L - \varphi_2^* L^2$) are not equal and less than unity in modulus, and regardless of their real or complex nature, the limit of $c_{1,h}$ is given by:

$$\bar{c}_1 = -\frac{1}{(1-\lambda_1)(1-\lambda_2)} ;$$

these conditions can equivalently be expressed in the following way : $\varphi_1^* + \varphi_2^* < 1$, $\varphi_2^* - \varphi_1^* < 1$ and $|\varphi_2^*| < 1$. These are exactly the stationarity conditions of the Gaussian AR(2) process x_{t+1} under the risk-neutral probability $\mathbb{Q}$.

If we substitute $\overline{c}_1$ into the second equation of system (1) we find, consequently, the limit of $c_{2,h}$:

$$\overline{c}_2 = -\varphi_2^* \frac{1}{(1-\lambda_1)(1-\lambda_2)} \, .$$

The recursive equation characterizing d_h is given by:

$$d_h = \begin{cases} 0 & \text{for } h = 1\,, \\ \nu^* \sum_{j=1}^{h-1} c_j + \frac{1}{2}\sigma^2 \sum_{j=1}^{h-1} c_j^2\,, & \forall h \geq 2\,, \end{cases}$$

that is, it is a function of (some parameter and) c_j for $j \in \{1, \ldots, h-1\}$.

Exercise N° 07.

Let us consider again the system of difference equations (c_h, d_h):

$$\begin{cases} c_h &= \Phi^{*'} c_{h-1} - \alpha \\ d_h &= -\beta + c_{1,h-1}\nu^* + \frac{1}{2}c_{1,h-1}^2\sigma^2 + d_{h-1}\,, \end{cases}$$

with initial conditions $c_0 = 0$ and $d_0 = 0$. Let us consider in that exercise the general case where x_{t+1} follows a Gaussian AR(p) process. In this case, it is well known that the steady state $\overline{\mathbf{C}} = [\,\overline{c}_1, \ldots, \overline{c}_p\,]'$ of the system c_h is given, I denoting the $(p \times p)$ identity matrix, by:

$$\overline{\mathbf{C}} = -(I - \Phi^{*'})^{-1}\alpha\,, \tag{3}$$

under the (stability) condition that the p eigenvalues $(\lambda_1, \ldots, \lambda_p)$ of $\Phi^{*'}$ are all smaller than unity in modulus, or, equivalently, that the risk-neutral dynamics of (x_t) is stationary, or that the roots of the risk-neutral autoregressive polynomial (of degree p) $\Psi^*(L) = 1 - \varphi_1^* L - \ldots - \varphi_p^* L^p$ have a modulus larger than one (given that these roots are the inverse of the eigenvalues). More precisely, the system of equations c_h can be rewritten as:

$$\begin{cases} c_{1,h} &= \varphi_1^* c_{1,h-1} + c_{2,h-1} - \alpha_1 \\ c_{2,h} &= \varphi_2^* c_{1,h-1} + c_{3,h-1} - \alpha_2 \\ &\vdots \\ c_{p-1,h} &= \varphi_{p-1}^* c_{1,h-1} + c_{p,h-1} - \alpha_{p-1} \\ c_{p,h} &= \varphi_p^* c_{1,h-1} - \alpha_p\,, \end{cases}$$

and if we substitute the p^{th} equation in the $(p-1)^{th}$ for $c_{p,h-1}$, and then the $(p-1)^{th}$ equation in the $(p-2)^{th}$ for $c_{p-1,h-1}$, and so on till the first one, we find that $c_{1,h}$ is described by the following p^{th} order linear difference equation :

$$\Psi^*(L)c_{1,h} = -\sum_{i=1}^{p} \alpha_i\,,$$

where $\Psi^*(L) = 1 - \varphi_1^* L - \ldots - \varphi_p^* L^p$ operates here to h. The remaining equations are given by :

$$c_{p-j,h} = -\sum_{i=0}^{j} \alpha_{p-i} + \sum_{i=0}^{j} \varphi_{p-i}^* c_{1,h-j+i-1}\,, \quad j \in \{0, \ldots, p-2\}\,.$$

Given the risk-neutral stationary assumption on the process (x_t), the relations $c_{p-j,h}$, for $j \in \{0, \ldots, p-1\}$, converge at an exponential rate with possible oscillations, when $h \to \infty$. The limits are :

$$\begin{aligned} \bar{c}_1 &= -\frac{\sum_{i=1}^{p} \alpha_i}{\Psi^*(1)}\,, \\ \bar{c}_{p-j} &= -\sum_{i=0}^{j} \alpha_{p-i} + \bar{c}_1 \sum_{i=0}^{j} \varphi_{p-i}^*\,, \quad j \in \{0, \ldots, p-2\}\,. \end{aligned}$$

Note that $\Psi^*(1) > 0$ because of the stability conditions.

With regard to d_h, its equation gives the specification of the long-term yield $R(t, \infty)$ as a function of the steady state $\bar{c}_1$. Indeed, the difference equation d_h can be written (assuming the identification condition $\beta = 0$, as we have seen in Exercise N° 02) as:

$$d_h = \begin{cases} 0 & \text{for } h = 1\,, \\ \nu^* \sum_{j=1}^{h-1} c_{1,j} + \frac{1}{2}\sigma^2 \sum_{j=1}^{h-1} c_{1,j}^2\,, & \forall h \geq 2\,, \end{cases} \tag{4}$$

and, under the stability of the system c_h, we have from the yield-to-maturity formula that:

$$\begin{aligned} R(t, \infty) &= \lim_{h \to +\infty} R(t, h) \\ &= \lim_{h \to +\infty} -\frac{c_h{}'}{h} X_t - \frac{\nu^*}{h} \sum_{j=1}^{h-1} c_{1,j} - \frac{\sigma^2}{2h} \sum_{j=1}^{h-1} c_{1,j}^2 = -\bar{c}_1 \nu^* - \frac{1}{2} (\bar{c}_1 \sigma)^2\,, \end{aligned}$$

which is positive under the condition $\left[\nu^* + \frac{1}{2}\sigma^2 \bar{c}_1\right] > 0$.

The shape of c_h (and thus the yield curve shapes), for h varying, depends on whether the eigenvalues $(\lambda_1, \ldots, \lambda_p)$ of $\Phi^{*'}$ are real or complex, single or multiple, larger or smaller than one in modulus.

Exercise N° 08 [Gaussian AR(p) Factor Dynamics under the S-Forward probability].

The S-forward Laplace transform of x_{t+1}, conditionally to $I_t = (\underline{x_t})$, can be written in the following way:

$$\begin{aligned} & E_t^{\mathbb{Q}^{(S)}}[\exp(u x_{t+1})] \\ = \quad & \frac{1}{B(t,S)} E_t^{\mathbb{Q}} \left[\exp(-r_t - \ldots - r_{S-1} + u x_{t+1})\right] . \end{aligned} \tag{5}$$

Now, starting from the identity

$$\log[B(t,T)] = \textstyle\sum_{j=1}^{t} \rho(j,T) + \log[B(0,T)] \,, \tag{6}$$

we have in the risk-neutral world:

$$\log[B(t,T)] \quad = \quad -\textstyle\sum_{j=1}^{t} \omega(j,T)\eta_j + \sum_{j=1}^{t} r_{j-1} - \frac{1}{2}\sum_{j=1}^{t} \omega(j,T)^2 + \log[B(0,T)] \,. \tag{7}$$

If we put $T = t$ in (7), we get a relation for the sum of the short-rates:

$$\textstyle\sum_{j=1}^{t} r_{j-1} \quad = \quad \sum_{j=1}^{t} \omega(j,t)\eta_j + \frac{1}{2}\sum_{j=1}^{t} \omega(j,t)^2 - \log[B(0,t)] \,, \tag{8}$$

that we can substitute in (7) to find the following alternative representation for the bond price process:

Proposition : For every fixed maturity T, the zero-coupon bond price process $B(\cdot,T) = [\,B(t,T),\, 0 \leq t \leq T\,]$, under the risk-neutral probability $\mathbb{Q}$, can be written as :

$$B(t,T) \quad = \quad \tfrac{B(0,T)}{B(0,t)} \exp\left(-\textstyle\sum_{j=1}^{t} [\omega(j,T) - \omega(j,t)]\,\eta_j - \frac{1}{2}\sum_{j=1}^{t} [\omega(j,T)^2 - \omega(j,t)^2]\right) . \tag{9}$$

From relation (8), we have that, under the risk-neutral measure $\mathbb{Q}$, the sum of short-term rates in the above formula can be written as:

$$\begin{aligned} \sum_{j=t+1}^{S} r_{j-1} \quad &= \quad \sum_{j=1}^{S} r_{j-1} - \sum_{j=1}^{t} r_{j-1} \\ &= \quad \sum_{j=1}^{S} \omega(j,S)\eta_j - \sum_{j=1}^{t} \omega(j,t)\eta_j \\ &\qquad + \frac{1}{2}\left[\sum_{j=1}^{S} \omega(j,S)^2 - \sum_{j=1}^{t} \omega(j,t)^2\right] + \log\left[\frac{B(0,t)}{B(0,S)}\right] \end{aligned}$$

and, consequently, we get:

$$
\begin{aligned}
& E_t^{\mathbb{Q}^{(S)}}[\exp(ux_{t+1})] \\
= \quad & \frac{\exp\left[-\frac{1}{2}\left[\sum_{j=1}^{S}\omega(j,S)^2 - \sum_{j=1}^{t}\omega(j,t)^2\right] - \sum_{j=1}^{t}\left[\omega(j,S)-\omega(j,t)\right]\eta_j - \log\left[\frac{B(0,t)}{B(0,S)}\right]\right]}{B(t,S)} \times \\
& E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=t+2}^{S}\omega(j,S)\eta_j - \omega(t+1,S)\eta_{t+1} + u[\nu^* + \varphi^{*'}X_t + \sigma^*\eta_{t+1}]\right)\right] \\
= \quad & k_{t,S}E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=t+2}^{S}\omega(j,S)\eta_j\right)\right] \times \\
& \qquad E_t^{\mathbb{Q}}\left[\exp\Big(u(\nu^* + \varphi^{*'}X_t) + \ (u\sigma^* - \omega(t+1,S))\eta_{t+1}\Big)\right] \\
= \quad & k'_{t,S}\exp\left[u\left[\nu^* + \varphi^{*'}X_t - \sigma^*\omega(t+1,S)\right] + \tfrac{1}{2}u^2\sigma^{*2}\right] \ ;
\end{aligned} \tag{10}
$$

now, using (9) we have that

$$
\begin{aligned}
k_{t,S} \quad = \quad & \frac{\exp\left[-\frac{1}{2}\left[\sum_{j=1}^{S}\omega(j,S)^2 - \sum_{j=1}^{t}\omega(j,t)^2\right] - \sum_{j=1}^{t}\left[\omega(j,S)-\omega(j,t)\right]\eta_j - \log\left[\frac{B(0,t)}{B(0,S)}\right]\right]}{B(t,S)} \\
= \quad & \exp\left[-\frac{1}{2}\sum_{j=t+1}^{S}\omega(j,S)^2\right]
\end{aligned}
$$

and that

$$
k'_{t,S} \quad = \quad k_{t,S}\exp\left[\frac{1}{2}\sum_{j=t+1}^{S}\omega(j,S)^2\right] = 1\,.
$$

Consequently, we recognize the conditional Laplace transform of the following Gaussian AR(p) stochastic process:

$$
x_{t+1} = \nu_S + \varphi_1^* x_t \ + \ \ldots \ + \ \varphi_p^* x_{t+1-p} \ + \sigma^*\xi_{t+1}\,,
$$

with

$$
\nu_S = \nu^* - \sigma^*\omega(t+1,S)\,,
$$

and where $\xi_{t+1} \sim \mathcal{IIN}(0,1)$ under $\mathbb{Q}_S$.

Exercise N° 09 [Zero-Coupon Bond Return Process under the S-Forward probability].

We have that:

$$\begin{aligned}\rho(t+1,\,T) &= \log\left[B(t+1,\,T)\right] - \log\left[B(t,\,T)\right] \\ &= c'_{T-t-1}X_{t+1} + d_{T-t-1} - c'_{T-t}X_t - d_{T-t} \\ &= c'_{T-t-1}\left[X_{t+1} - \Phi^* X_t - \tilde{\nu}^*\right] + r_t - \tfrac{1}{2}c^2_{1,T-t-1}\sigma^{*2} \\ &= c'_{T-t-1}\left[\sigma^*\left(\tilde{\xi}_{t+1} - \omega(t+1,S)e_1\right)\right] + r_t - \tfrac{1}{2}c^2_{1,T-t-1}\sigma^{*2} \\ &= r_t + \omega(t+1,T)\omega(t+1,S) \\ &\qquad - \tfrac{1}{2}\omega(t+1,T)^2 - \omega(t+1,T)\xi_{t+1}\,,\end{aligned}$$

and the first part of the result is proved. Now, if we calculate $E_t^{\mathbb{Q}^{(S)}}\{\exp\left[\rho(t+1,\,T)\right]\}$ we have:

$$\begin{aligned}E_t^{\mathbb{Q}^{(S)}}\exp\left[\rho(t+1,\,T)\right] &= E_t^{\mathbb{Q}^{(S)}}\exp\left[r_t + \omega(t+1,T)\omega(t+1,S) - \tfrac{1}{2}\omega(t+1,T)^2 - \omega(t+1,T)\xi_{t+1}\right] \\ &= \exp[r_t + \omega(t+1,T)\omega(t+1,S) - \tfrac{1}{2}\omega(t+1,T)^2]\, E_t^{\mathbb{Q}^{(S)}}\exp[-\omega(t+1,T)\xi_{t+1}] \\ &= \exp(r_t + \omega(t+1,T)\omega(t+1,S))\end{aligned}$$

and therefore $\lambda_t^{\mathbb{Q}^{(S)}}(\rho,1) = \log E_t^{\mathbb{Q}^{(S)}}\{\exp\left[\rho(t+1,\,T)\right]\} - r_t = \omega(t+1,T)\omega(t+1,S)$.

Exercise N° 10 [No-arbitrage restrictions for the short rate and spread].

We have a bivariate Gaussian VAR(1) ATSM given by:

$$\begin{aligned}x_{t+1} &= \nu + \Phi x_t + \Sigma\varepsilon_{t+1}\,, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, I_2) \text{ (under } \mathbb{P}) \\ M_{t,t+1} &= \exp\left[-\beta - \alpha' x_t + \Gamma_t'\varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t'\Gamma_t\right]\,, \quad \text{(SDF)} \\ \Gamma_t &= \Gamma(x_t) = (\gamma_o + \gamma\, x_t)\,, \\ R(t,h) &= -\frac{C_h{}'}{h}x_t - \frac{D_h}{h}\,, \\ C_h &= -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} = -\alpha + \Phi^{*\prime} C_{h-1}\,, \\ D_h &= -\beta + C'_{h-1}(\nu + \Sigma\gamma_o) + \tfrac{1}{2}C'_{h-1}(\Sigma\Sigma')C_{h-1} + D_{h-1}\,, \\ & C_0 = 0, D_0 = 0\,,\end{aligned}$$

where $x_t = (r_t, S_t)'$, with $r_t = R(t,t+1)$ the yield with the shortest maturity in our data base (it is the short rate) and $S_t = R_t - r_t$ the spread between the long rate (the yield with the longest maturity in our data base) and the short rate.

I have to impose no-arbitrage restrictions on both components of the factor (x_t) given that they contains yields at different maturities.
First, I have to impose that $R(t, t+1) = r_t$: this condition generates the no-arbitrage restriction $R(t, 1) = \beta + \alpha' x_t = \beta + \alpha_1 r_t + \alpha_2 S_t = r_t$. Clearly, $r_t = \beta + \alpha' x_t = \beta + \alpha_1 r_t + \alpha_2 S_t$ if and only if $\beta = 0$, $\alpha_1 = 1$ and $\alpha_2 = 0$. These conditions are equivalent to $C_1 = -(1, 0)$ and $D_1 = 0$.
Second, let us denote by H the longest maturity in our data base. I have to impose that $R(t, t+H) = R_t$ for any t. In this case we have:

$$
\begin{aligned}
& -\frac{1}{H}[C_{1,H}\, r_t + C_{2,H}\, S_t + D_H] = R_t \\
& \Leftrightarrow C_{1,H}\, r_t + C_{2,H}\,(R_t - r_t) + D_H = -H R_t \\
& \Leftrightarrow [C_{1,H} - C_{2,H}]\, r_t + C_{2,H}\, R_t + D_H = -H R_t \\
& \Leftrightarrow C_{1,H} = C_{2,H}\,, \;\; C_{2,H} = -H\,, \;\; D_H = 0\,,
\end{aligned}
$$

that is $C_H = -H\,(1, 1)'$ and $D_H = 0$.