Fixed Income and Credit Risk

Lecture 4

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Lecture 4 - Part I

Discrete-Time Univariate Gaussian

AR(p) Term Structure Models

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4.1 Univariate Gaussian AR(1) Factor-Based Term Structure Models

4.1.1 Historical Dynamics

- ☐ We consider an economy, in a **dynamic discrete-time setting**, between dates 0 and T.

- ☐ The new information in the economy at date t is denoted by x_t and the overall information at date t is $\underline{x}_t = (x_t, x_{t-1}, ..., x_0)$. It is the (common) **information judged relevant** by each investor to price assets.

- ☐ x_t is called a **factor or a state vector**, and it may be observable, partially observable or unobservable by the econometrician. The size of x_t is K.

- $x_t =$ **observable**

  $\rightarrow$ interest rates of different maturities, inflation rate, gross domestic product, ...

- $x_t =$ **non observable**

  $\rightarrow$ level, slope and curvature factors, market regimes (using regime-switching models), stochastic volatility, jumps (market crashes), ...

- $x_t =$ **partially observable**

  $\rightarrow$ $x_t = (x_{1,t}, x_{2,t})'$ where $x_{1,t}$ is observable and $x_{2,t}$ is not.

- The **historical dynamics** of x_t is defined by the joint distribution of $\underline{x}_T$, denoted by $\mathbb{P}$, or by the conditional probability density function (p.d.f.):

$$f_t(x_{t+1}|\underline{x}_t)\,,$$

- or by the **conditional Laplace transform** (L.T.):

$$\varphi_t(u|\underline{x}_t) = \varphi_t(u) = E[\exp(u'x_{t+1})|\underline{x}_t] = E_t[\exp(u'x_{t+1})]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^K$ (containing zero).

- We also introduce the **conditional Log-Laplace transform**:

$$\psi_t(u|\underline{x}_t) = \psi_t(u) = \text{Log}[\varphi_t(u|\underline{x}_t)]\,.$$

□ Let us assume that $K = 1$ and that the (non observable) factor x_{t+1} is a Gaussian AR(1) process of the following type:

$$x_{t+1} \;=\; \nu + \varphi\, x_t + \sigma\varepsilon_{t+1}\,,$$

where ε_{t+1} is a Gaussian white noise with $\mathcal{N}(0,1)$ distribution.

□ $E_t[x_{t+1}] = \nu + \varphi x_t$ and $V_t[x_{t+1}] = \sigma^2$, $\Rightarrow \quad x_{t+1} \,|\, x_t \sim N(\nu + \varphi x_t, \sigma^2)$

and $x^e_{t+k\,|\,t} := E_t[x_{t+k}] = (1 + \varphi + \ldots + \varphi^{k-1})\,\nu + \varphi^k\, x_t$ (under $\mathbb{P}$).

□ Under stationarity (i.e., $|\,\varphi\,| < 1$), we have $E[x_t] = \dfrac{\nu}{1-\varphi}$ and $V[x_t] = \dfrac{\sigma^2}{1-\varphi^2}$,

$\Rightarrow\; x_t \sim N\left(\dfrac{\nu}{1-\varphi}, \dfrac{\sigma^2}{1-\varphi^2}\right)$, with $\lim_{k\to+\infty} E_t[x_{t+k}] = E[x_t]$ (under $\mathbb{P}$).

□ Let us remember that the Laplace transform of a scalar Gaussian random variable

$Y \sim N(\mu, \omega^2)$ is:

$$\varphi(u) = E[\exp(uY)] = \exp\left(u\mu + \frac{1}{2}\,u^2\omega^2\right) .$$

□ This means that:

$$\varphi_t(u|\underline{x}_t) = \varphi_t(u) = \exp\left[u(\nu\, +\, \varphi x_t) + \frac{1}{2}\,u^2\,\sigma^2\right] ,$$

□ and

$$E[\exp(ux_t)] = \exp\left[u\left(\frac{\nu}{1-\varphi}\right) + \frac{1}{2}\,u^2\,\frac{\sigma^2}{1-\varphi^2}\right] .$$

4.1.2 The Stochastic Discount Factor

- We price assets (ZCBs in our case!) following the no-arbitrage principle.

- We are in a incomplete market setting and therefore, under AAO, we have an infinitely many positive SDFs.

- The development of the zero-coupon bond (no arbitrage) pricing model is characterized:

  - after the historical distribution assumption (presented above),

  - by the parametric specification of a positive stochastic discount factor (SDF) $M_{t,t+1}$, for the period $(t, t+1)$.

☐ The price $y(t)$ at t of a financial asset (basic asset, derivative, ...) paying $y(T)$ at T is:

$$y(t) \;=\; E\left[M_{t,t+1} \cdot \ldots \cdot M_{T-1,T}\, y(T) \,|\, \underline{x}_t\right] = E_t\left[M_{t,T}\, y(T)\right] \; .$$

☐ We choose a SDF which is exponential-affine in the state variable x_{t+1}, that is (equivalently), in its noise ε_{t+1} :

$$M_{t,t+1} = \exp\left[-\beta - \alpha\, x_t + \Gamma_t\, \varepsilon_{t+1} - \frac{1}{2}\,\Gamma_t^2\right] \; :$$

– the coefficients α and β are path independent (constant!);

– $\Gamma_t = \Gamma(x_t) = (\gamma_o + \gamma\, x_t)$ is a stochastic risk correction coefficient, also called **Market Price of Factor Risk** [see following sections].

- Now, the absence of arbitrage restriction on the ZCB with unitary residual maturity requires:

$$E_t(M_{t,t+1}) = \mathsf{exp}(-r_t),$$

where r_t is the (predetermined) short-term interest rate between t and $t+1$.

- This condition implies the relation $r_t = \beta + \alpha x_t$.

- This means that, under the absence of arbitrage opportunities, the SDF can be written as:

$$M_{t,t+1} = \mathsf{exp}\left[-r_t + \Gamma_t\,\varepsilon_{t+1} - \frac{1}{2}\,\Gamma_t^2\right] = \mathsf{exp}(-r_t)\,\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}}\,.$$

4.1.3 The Risk Premium

- In order to give an interpretation of the risk-correction coefficient Γ_t, we consider the following definition of risk premium [see also Dai, Singleton and Yang (2007, RFS)]:

Definition 1 : If we denote by P_t the price at time t of a given asset, its risk premium between t and $t+1$ is:

$$\lambda_t \quad = \quad \log E_t \left(\frac{P_{t+1}}{P_t} \right) - r_t = \log E_t \exp(y_{t+1}) - r_t \, ,$$

where $y_{t+1} = \log(P_{t+1}/P_t)$ denotes the one-period geometric return of the asset.

☐ We can interpret λ_t as the excess growth rate of the expected price with respect to the present price.

☐ Now, starting from this definition of the risk premium we obtain interpretations of the function Γ_t, appearing in the SDF, by means of the following example.

☐ **Example :** If we consider an asset providing the payoff $\exp(-bx_{t+1})$ at $t+1$, its price in t is given by:

$$\begin{aligned} P_t &= E_t\left[M_{t,t+1}P_{t+1}\right] = E_t\left[\exp\left(-r_t - \frac{1}{2}\Gamma_t^2 + (\Gamma_t - b\sigma)\varepsilon_{t+1} - b(\nu + \varphi x_t)\right)\right] \\ &= \exp\left[-r_t - b(\nu + \varphi x_t) - b\sigma\Gamma_t + \frac{1}{2}(b\sigma)^2\right] , \end{aligned}$$

□ and

$$
\begin{aligned}
E_t P_{t+1} &= E_t[\exp(-b x_{t+1})] = \exp\left[-b(\nu + \varphi x_t)\right] E_t \left\{\exp\left[-b\sigma\varepsilon_{t+1}\right]\right\} \\
&= \exp\left[-b(\nu + \varphi x_t) + \tfrac{1}{2}(b\sigma)^2\right] .
\end{aligned}
$$

□ Finally, the risk premium is:

$$
\lambda_t = b\,\sigma\,\Gamma_t\,.
$$

□ Therefore, b, Γ_t and σ can be seen respectively as a risk sensitivity of the asset, a risk price and a risk measure.

4.1.4 The Affine Term Structure of Interest Rates

□ With the specification of the SDF, we determine the price of a zero-coupon bond in the following way :

$$B(t, t+h) = E_t\left[M_{t,t+1} \cdot \ldots \cdot M_{t+h-1,t+h}\right] ,$$

where $B(t, t+h)$ denotes the price at time t for a ZCB with residual maturity h.

□ **Proposition 1 :** The price at date t of the zero-coupon bond with residual maturity h is:

$$B(t, t+h) = \exp(c_h x_t + d_h)\,, \;\; h \geq 1\,,$$

☐ where c_h and d_h satisfies the recursive equations:

$$\begin{cases} c_h &= -\alpha + \varphi^* c_{h-1}\,, \\ d_h &= -\beta + c_{h-1}\nu^* + \frac{1}{2}c_{h-1}^2\sigma^2 + d_{h-1}\,, \end{cases}$$

with $\varphi^* = (\varphi + \sigma\gamma)$, $\nu^* = (\nu + \gamma_o\sigma)$ [keep in mind these parameters].

☐ The initial conditions of the recursive (difference) equations are:

- at $h = 0$ we have $B(t,t) = 1$, implying the conditions $c_0 = 0$ and $d_0 = 0$.
- or, at $h = 1$ we have $B(t, t+1) = \exp(-r_t)$, implying the conditions $c_1 = -\alpha$ and $d_1 = -\beta$.

☐ **Proof of Proposition 1 :** given that $M_{t,t+1}$ is exponential-affine in ε_{t+1} (i.e. x_{t+1}) and that the conditional Laplace transform of x_{t+1} is exponential-affine in the conditioning variable (x_t) we suggest that the ZCB pricing formula at date t be an exponential-affine function of x_t and then "we check if it works".

☐ We proceed in the following way: a) we suggest $B(t, t+h) = \exp(c_h\, x_t + d_h)$ and we (equivalently) rewrite the pricing formula in terms of the payoff $B(t+1, t+h) = \exp(c_{h-1}\, x_{t+1} + d_{h-1})$:

$$
\begin{aligned}
B(t, t+h) &= \exp(c_h\, x_t + d_h) \\
&= E_t[M_{t,t+1} \cdots M_{t+H-1,t+H}] \\
&= E_t[M_{t,t+1} B(t+1, t+h)] \\
&= E_t\left[\exp\left(-\beta - \alpha\, x_t + \Gamma_t\, \varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t^2\right) \exp(c_{h-1}\, x_{t+1} + d_{h-1})\right] ,
\end{aligned}
$$

□ b) we do the algebra (calculating the conditional Laplace transform) obtaining:

$$
\begin{aligned}
& B(t,t+h) \\
= \;& \exp(c_h\, x_t + d_h) \\
= \;& \exp\left[-\beta - \alpha x_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1}\right] \times E_t[\exp\left(\Gamma_t \varepsilon_{t+1} + c_{h-1} x_{t+1}\right)] \\
= \;& \exp\left[-\beta - \alpha x_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1} + c'_{h-1}(\varphi x_t + \nu)\right] \times \; E_t[\exp\left(\Gamma_t + \sigma c_{h-1}\right)\varepsilon_{t+1})] \\
= \;& \exp\left[(-\alpha + \varphi c_{h-1} + c_{h-1}\sigma\gamma) x_t + (-\beta + c_{h-1}\nu + \tfrac{1}{2} c_{h-1}^2 \sigma^2 + \gamma_o c_{h-1}\sigma + d_{h-1})\right] ,
\end{aligned}
$$

□ c) and by identifying the coefficients we find the recursive relation presented in Proposition 1.

☐ The ZCB price at date t is an exponential-affine function of the factor (x_t) at the date $t \rightarrow$ it is function ONLY of the information at time t.

☐ **Corollary 1 :** The yields to maturity (continuously compounded spot rates) associated to the ZCB pricing formula are :

$$\begin{aligned} R(t, t+h) &= -\frac{1}{h} \log B(t, t+h) \\ &= -\frac{c_h}{h} x_t - \frac{d_h}{h}, \quad h \geq 1, \end{aligned}$$

and they are **affine functions** of the factor x_t.

- For a given t and with h varying, $R(t, t+h)$ is the so-called **affine term structure of interest rates**.

- For that reason the model is called **Affine Term Structure Model (ATSM)**.

- Given that the factor x_t is described by a discrete-time Gaussian stochastic process (the AR(1) process), then we talk about **Gaussian Discrete-Time ATSM**.

- x_t is a scalar process : **Univariate Gaussian ATSM**.

4.1.5 Excess Returns of Zero-Coupon Bonds

□ We have the following specification for the zero-coupon bond return process.

□ **Proposition 2 :** Under the absence of arbitrage opportunity, and for a fixed maturity T, the one-period geometric zero-coupon bond return process $\rho = [\,\rho(t,T), 0 \leq\, t \leq T\,]$, where $\rho(t+1,T) = \log[B(t+1,\, T)] - \log[B(t,\, T)]$, is given by:

$$\rho(t+1,\, T) \;\;=\;\; r_t - \tfrac{1}{2}\,\omega(t+1,\, T)^2 + \omega(t+1,\, T)\,\Gamma_t - \omega(t+1,\, T)\,\varepsilon_{t+1}\,,$$

where $\omega(t+1,\, T) = -(\sigma c_{T-t-1})$ [Proof : exercise].

□ This means that the process ρ is such that:

$$\rho(t+1,T)\,|\,\underline{x_t} \;\sim\; N\left[\mu(t+1,\,T),\omega(t+1,\,T)^2\right]$$

where $\mu(t+1,\,T) = r_t - \frac{1}{2}\,\omega(t+1,\,T)^2 + \omega(t+1,\,T)\Gamma_t\,,$

and $\omega(t+1,\,T)^2 = (\sigma c_{T-t-1})^2\,.$

□ The associated risk premium between t and $t+1$, denoted by $\lambda_t(T)$, is:

$$\lambda_t(T) = \log E_t \exp[\,\rho(t+1,\,T)] - r_t = \omega(t+1,\,T)\,\Gamma_t\,.$$

□ $\Gamma_t = (\gamma_o + \gamma x_t)$ plays (for any T) the role of a risk premium per unit of "risk" $\omega(t+1,\,T)$.

- In particular, for a fixed $\gamma \neq 0$, the variability of $\lambda_t(T)$ is driven by x_t.

- If we assume $\gamma = 0$ (i. e., $\Gamma_t = \gamma_o$), the risk correction coefficient and the risk premium of the zero-coupon bond become constants.

- Also note that, if $T = t + 2$ and $x_t = r_t$, we have $\omega(t+1, T) = \sigma$ and we get the result of the example presented in Section 4.1.3 for $b = 1$.

- We will see during the next Lecture that this property of the excess bond return process gives the opportunity to easily estimate the model, and in particular (γ_o, γ).

4.1.6 Risk-Neutral Dynamics

- In the previous sections we have presented the Gaussian AR(1) Factor-Based Term Structure Model under the historical probability $\mathbb{P}$.

- Under the absence of arbitrage opportunity, there exist a probability $\mathbb{Q} \sim \mathbb{P}$ under which asset prices, evaluated with respect to some numeraire N_t, are martingales:

$$\frac{y(t)}{N_t} = E_t^{\mathbb{Q}}\left[\frac{y(t+1)}{N_{t+1}}\right] .$$

- $\mathbb{Q}$ is be the probability (equivalent to $\mathbb{P}$) defined by the sequence of conditional densities:

$$\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{N_{t+1}\, M_{t,t+1}}{N_t} > 0 \, , \;\; E_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}}\right] = 1 \, , t \in \{0, \ldots, T-1\} \, .$$

- □ The most used choices of numeraire are the money-market account (we are going to use) and the ZCB choice (presented in one of the following sections).

- □ If we consider as numeraire the money-market account $N_t = \exp(r_0 + \ldots + r_{t-1}) = A_{0,t}$, where $(A_{0,t})^{-1} = E_0(M_{0,1}) \cdots E_{t-1}(M_{t-1,t})$, the associated equivalent probability $\mathbb{Q}$ has a one-period conditional density, with respect to $\mathbb{P}$, given by :

$$\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{A_{0,t+1} M_{t,t+1}}{A_{0,t}} = \frac{M_{t,t+1}}{E_t(M_{t,t+1})} = e^{r_t}\, M_{t,t+1} \, .$$

  and it is called **risk-neutral probability measure**.

□ This means that the pricing formula $y(t) = E_t[M_{t,t+1}y(t+1)]$ can be written:

$$\begin{aligned} y(t) &= E_t\left[\frac{M_{t,t+1}}{E_t[M_{t,t+1}]}E_t[M_{t,t+1}]y(t+1)\right] \\ &= E_t^{\mathbb{Q}}[\exp(-r_t)y(t+1)]\,. \end{aligned}$$

□ In a general $(T-t)$-period horizon, the conditional (to $\underline{x_t}$) density of the risk-neutral probability $\mathbb{Q}$ with respect to the historical probability $\mathbb{P}$ is given by:

$$\begin{aligned} \frac{d\mathbb{Q}_{t,T}}{d\mathbb{P}_{t,T}} &= \frac{M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}}{E_t(M_{t,t+1})\cdot\ldots\cdot E_{T-1}(M_{T-1,T})} \\ &= \exp(r_t+\ldots+r_{T-1})\,M_{t,T}\,, \end{aligned}$$

□ This means that, for any payoff $y(T)$ at T, we have :

$$y(t) = E_t^{\mathbb{Q}}[\exp(-r_t-\ldots-r_{T-1})y(T)]\,,$$

and $y(t)/A_{0,t}$ is a $\mathbb{Q}$-martingale.

- The one-period transition from the historical world to the risk-neutral one is given, in our model, by the conditional density function :

$$\frac{M_{t,t+1}}{E_t(M_{t,t+1})} = \exp\left[\Gamma_t\,\varepsilon_{t+1} - \frac{1}{2}\,\Gamma_t^2\right] .$$

- Moreover, for any asset, the price P_t at t is equal to $\exp(-r_t)E_t^{\mathbb{Q}}(P_{t+1})$ and, therefore, the risk premium λ_t presented in Definition 1 is equal to:

$$\lambda_t = \log E_t(P_{t+1}) - \log E_t^{\mathbb{Q}}(P_{t+1})\,,$$

□ The risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$
\begin{aligned}
E_t^{\mathbb{Q}}[\exp(ux_{t+1})] = E_t\left[\frac{M_{t,t+1}}{E_t(M_{t,t+1})}\exp(ux_{t+1})\right] & \\
= \quad & E_t\left[\exp\left((\gamma_o+\gamma x_t)\,\varepsilon_{t+1}-\tfrac{1}{2}\,(\gamma_o+\gamma x_t)^2+ux_{t+1}\right)\right] \\
= \quad & \exp\left[u(\nu+\varphi x_t)-\tfrac{1}{2}(\gamma_o+\gamma x_t)^2\right]\;\times\;E_t[\exp(\gamma_o+\gamma x_t+u\sigma)\varepsilon_{t+1}] \\
= \quad & \exp\left[u[(\nu+\sigma\gamma_o)+(\varphi+\sigma\gamma)x_t]+\tfrac{1}{2}u^2\sigma^2\right] \\
= \quad & \exp\left[u(\nu^*+\varphi^* x_t)+\tfrac{1}{2}u^2\sigma^2\right]\;,
\end{aligned}
$$

□ **Proposition 3 :** Under the risk-neutral probability $\mathbb{Q}$, x_{t+1} is an AR(1) process of the following type:

$$x_{t+1} \;=\; \nu^* + \varphi^* x_t + \sigma^* \eta_{t+1}\,,$$

□ with

$$\nu^* \;=\; (\nu + \sigma\gamma_o)\,,\;\; \varphi^* = (\varphi + \sigma\gamma)\,,\;\; \sigma^* = \sigma\,.$$

and where $\eta_{t+1} \overset{\mathbb{Q}}{\sim} \mathcal{IIN}(0,1)$. Note that $\varepsilon_{t+1} = \eta_{t+1} + \Gamma_t$.

☐ If $\Gamma_t = \gamma_o$ (constant market price of risk), only the constant term changes.

If $\Gamma_t = 0$, then (x_t) has the same distribution under $\mathbb{P}$ and $\mathbb{Q}$.

☐ Indeed, if $\Gamma_t = 0$ we have $M_{t,t+1} = \exp(-r_t)$ and any payoff is discounted under $\mathbb{P}$ by the risk-free rate:

$$B(t, t+h) = E_t[\exp(-r_t - \ldots - r_{t+h-1})]\,, \text{ with } r_t = \beta + \alpha x_t\,.$$

☐ Meaning $\to$ assuming $M_{t,t+1} = \exp(-r_t)$ implies that we do not consider the factor (x_t) as a source of risk, additional to (different from) (r_t), affecting the ZCB price process.

☐ Indeed, in that case we have $\lambda_t(T) = \log E_t \exp[\,\rho(t+1, T)] - r_t = 0$.

□ **Proposition 4 :** In the risk-neutral framework, for a fixed maturity T, the one-period geometric zero-coupon bond return process satisfies the relation:

$$\rho(t+1,T) = r_t - \frac{1}{2}\,\omega(t+1,T)^2 - \omega(t+1,T)\,\eta_{t+1}\,,$$

□ with a risk premium equal to :

$$\lambda_t^{\mathbb{Q}}(T) = \log E_t^{\mathbb{Q}} \exp\left[\rho(t+1,T)\right] - r_t = 0\,.$$

4.1.7 The Gaussian short-rate model

- In what we have presented above, the factor x_t was latent. In the term structure literature several models are specified assuming $x_t = r_t$.

- The shape and the dynamics of the (ENTIRE!) yield curve is driven (ONLY!) by the short-rate process.

- It is convenient to have observable factors: we can specify the historical dynamics of the factor starting from the observed stylized facts (autocorrelation, marginal moments, mean-reversion, stationarity, ...) on the short-rate.

□ We assume that the factor $x_{t+1} = r_{t+1}$ is a Gaussian AR(1) process of the following type:

$$r_{t+1} = \nu + \varphi r_t + \sigma \varepsilon_{t+1} ,$$

□ we have the same SDF, but now the AAO condition $E_t(M_{t,t+1}) = \exp(-r_t)$ implies $\beta = 0$ and $\alpha = 1$. We have to guarantee that the theoretical formula $R(t,h)$ generates, when $h = 1$, exactly the short rate process we have assumed under $\mathbb{P}$.

□ Clearly, under $\mathbb{Q}$ we have:

$$r_{t+1} = \nu^* + \varphi^* r_t + \sigma \eta_{t+1} ,$$

□ It is the discrete-time equivalent of the continuous-time Vasicek (1977) model.

- An interesting interpretation of Γ_t stands out when we write $R(t,h)$ for $h = 2$. It is easy to verify that:

$$R(t, t+2) = \frac{1}{2}\left[r_t + E_t(r_{t+1}) + \sigma\Gamma_t - \frac{1}{2}\sigma^2\right],$$

- The term $\frac{1}{2}\left[r_t + E_t(r_{t+1})\right]$ is the average sequence of future short rates ($\rightarrow$ Expectation Hypothesis Theory!), while $(\sigma^2/2)$ is a Jensen inequality term ($E[\exp(X)] > \exp[E(X)]$).

- The term $\frac{1}{2}\sigma\Gamma_t$ is the non-zero time-varying Term Premia: if $\Gamma_t = \gamma_o$ then TP is constant over time and depend only on the residual maturity (EH). If $\Gamma_t = 0$, then TP $= 0$ (PEH).

4.1.8 The S-Forward Dynamics

☐ In many financial applications, a convenient numeraire is the zero-coupon bond whose maturity S is the same as the derivative product we would like to price.

☐ More precisely, the equivalent martingale measure is determined in this case, for every date $t \in [0, S]$, by the numeraire:

$$N_t = \frac{B(t,S)}{B(0,S)},$$

and it is referred to as S-**forward probability** and denoted by $\mathbb{Q}^{(S)}$.

□ The one-period conditional (to $\underline{x_t}$) density of the S-forward probability $\mathbb{Q}^{(S)}$, with respect to the historical probability $\mathbb{P}$, is given by:

$$\frac{d\mathbb{Q}^{(S)}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{M_{t,t+1}B(t+1,S)}{B(t,S)},$$

□ while, the one-period conditional (again, to $\underline{x_t}$) density of the S-forward probability $\mathbb{Q}^{(S)}$ with respect to the risk-neutral probability $\mathbb{Q}$, is given by:

$$\frac{d\mathbb{Q}^{(S)}_{t,t+1}}{d\mathbb{Q}_{t,t+1}} = \frac{d\mathbb{Q}^{(S)}_{t,t+1}}{d\mathbb{P}_{t,t+1}} \frac{d\mathbb{P}_{t,t+1}}{d\mathbb{Q}_{t,t+1}} = E_t(M_{t,t+1})\frac{B(t+1,S)}{B(t,S)} = \exp(-r_t)\frac{B(t+1,S)}{B(t,S)}.$$

□ Therefore, in a $(T-t)$-period horizon (where $T \leq S$), the S-forward probability $\mathbb{Q}^{(S)}$ has a (conditional to $\underline{x_t}$) joint density with respect to the risk-neutral probability $\mathbb{Q}$ given by:

$$\frac{d\mathbb{Q}^{(S)}_{t,T}}{d\mathbb{Q}_{t,T}} = \prod_{\tau=t}^{T-1} \exp(-r_\tau) \frac{B(\tau+1,S)}{B(\tau,S)} = \frac{B(T,S)}{B(t,S)} \exp(-r_t - \ldots - r_{T-1}),$$

□ and the pricing formula of $y(T)$, for $S=T$, takes the following useful representation:

$$\begin{aligned} y(t) &= E_t^{\mathbb{Q}} [\exp(-r_t - \ldots - r_{T-1}) y(T))] \\ &= B(t,T) E_t^{\mathbb{Q}^{(T)}} [y(T)], \end{aligned}$$

in which the problem of derivative pricing reduces to calculating an expectation of the payoff $y(T)$.

□ The S-forward dynamics of x_{t+1} has an AR(1) representation of the following type:

$$x_{t+1} = \nu_S + \varphi^* x_t + \sigma^* \xi_{t+1} \, ,$$

□ with

$$\nu_S = \nu^* - \sigma^* \omega(t+1, S) \, ,$$

□ and where $\xi_{t+1} \sim \mathcal{IIN}(0,1)$ under $\mathbb{Q}^{(S)}$ [Proof : exercise]. Observe that $\varepsilon_{t+1} = \xi_{t+1} - \omega(t+1, S) + \Gamma_t$.

□ In the S-forward framework, the one-period geometric zero-coupon bond return process is described by the relation:

$$\rho(t+1,T) \;=\; -\omega(t+1,T)\,\xi_{t+1} + r_t - \tfrac{1}{2}\,\omega(t+1,T)^2 + \omega(t+1,T)\omega(t+1,S)\,,$$

□ with a one-period risk premium given by :

$$\lambda_t^{\mathbb{Q}^{(S)}}(T) = \log E_t^{\mathbb{Q}^{(S)}} \exp\left[\rho(t+1,T)\right] - r_t = \omega(t+1,T)\omega(t+1,S)\,,$$

[Proof : exercise].

□ Consequently, under the T-forward probability, the one-period risk premium per unit of $\omega(t+1,T)$ is given by the $\omega(t+1,T)$ itself.

4.2 Univariate Gaussian AR(p) Factor-Based Term Structure Models

4.2.1 Historical Dynamics

- We assume that the (scalar) exogenous factor x_{t+1} characterizing the specification of the term structure is an AR(p) process of the following type:

$$\begin{aligned} x_{t+1} &= \nu + \varphi_1 x_t + \ldots + \varphi_p x_{t+1-p} + \sigma\varepsilon_{t+1} \\ &= \nu + \varphi' X_t + \sigma\varepsilon_{t+1}\,, \end{aligned}$$

- where ε_{t+1} is a gaussian white noise with $\mathcal{N}(0,1)$ distribution.

- We have: $\varphi = [\varphi_1, \ldots, \varphi_p]'$, $X_t = [\,x_t, \ldots, x_{t+1-p}\,]'$, and where $\sigma > 0$, ν and φ_i, for $i \in \{1, \ldots, p\}$, are scalar coefficients.

- $E_t[x_{t+1}] = \nu + \varphi' X_t$ and $V_t[x_{t+1}] = \sigma^2 \Rightarrow x_{t+1} \,|\, x_t \sim N(\nu + \varphi' X_t, \sigma^2)$ (under $\mathbb{P}$).

- Under stationarity (i.e. the roots of the equation $1 - \sum_{j=1}^{p} \varphi_j \, z^j = 0$ all lie outside the unit circle), we have $E[x_t] = \dfrac{\nu}{1 - \sum_{j=1}^{p} \varphi_j} = \mu_x$ and $V[x_t] = \sigma_x^2$ [see Hamilton (1994), Chapter 3],

  $\Rightarrow \; x_t \sim N\left(\mu_x, \sigma_x^2\right)$ (under $\mathbb{P}$).

- Forecasts can be recursively calculated in the following way:

$$x^e_{t+k\,|\,t} := E_t[x_{t+k}] = \nu + \varphi_1 \, E_t[x_{t+k-1}] + \varphi_2 \, E_t[x_{t+k-2}] + \ldots + \varphi_p \, E_t[x_{t+k-p}] \,,$$

  starting from $E_t[x_{t+1}] = \nu + \varphi_1 \, x_t + \varphi_2 \, x_{t-1} + \ldots + \varphi_p \, x_{t-p+1}$.

☐ The conditional Laplace transform is given by:

$$\varphi_t(u|\underline{x}_t) = \varphi_t(u) = \exp\left[u(\nu + \varphi' X_t) + \frac{1}{2}\, u^2\, \sigma^2\right] ,$$

☐ and the marginal one is:

$$E[\exp(u x_t)] = \exp\left[u\, \mu_x + \frac{1}{2}\, u^2\, \sigma_x^2\right] .$$

☐ The model can also be represented in the following multivariate AR(1) form :

$$X_{t+1} = \tilde{\nu} + \Phi X_t + \sigma \tilde{\varepsilon}_{t+1}\,,$$

☐ where $\tilde{\nu} = [\,\nu, 0, \ldots, 0\,]'$ and $\tilde{\varepsilon}_{t+1} = [\,\varepsilon_{t+1}, 0, \ldots, 0\,]'$ are p-dimensional vectors,

☐ and where

$$\Phi = \begin{bmatrix} \varphi_1 & \ldots & \ldots & \varphi_{p-1} & \varphi_p \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & 1 & 0 \end{bmatrix}$$

is a $(p \times p)$-matrix.

4.2.2 Stochastic Discount Factor

□ We specify the following SDF:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' X_t + \Gamma_t \varepsilon_{t+1} - \frac{1}{2}\Gamma_t^2\right],$$

□ where the coefficients $\alpha = [\alpha_1, \ldots, \alpha_p]'$ and β are path independent, and where

$\Gamma_t = \Gamma(X_t) = (\gamma_o + \gamma' X_t) = \gamma_o + \gamma_1 \, x_t + \ldots + \gamma_p \, x_{t-p+1}$.

□ The no-arbitrage restriction $E_t(M_{t,t+1}) = \exp(-r_t)$, implies the relation $r_t = \beta + \alpha' X_t$.

4.2.3 The Risk Premium

☐ Given the definition of risk premium introduced in Lecture 3 (Part III):

$$\lambda_t = \log E_t \left(\frac{P_{t+1}}{P_t} \right) - r_t = \log E_t \exp(y_{t+1}) - r_t \, ,$$

☐ and given the same payoff $\exp(-bx_{t+1})$ at $t+1$, its price in t is given by:

$$\begin{aligned} P_t &= E_t \left[M_{t,t+1} P_{t+1} \right] = \exp \left[-r_t - b(\nu + \varphi' X_t) - b\sigma \Gamma_t + \frac{1}{2}(b\sigma)^2 \right] , \\ E_t P_{t+1} &= E_t[\exp(-bx_{t+1})] = \exp \left[-b(\nu + \varphi' X_t) + \frac{1}{2}(b\sigma)^2 \right] . \end{aligned}$$

☐ the risk premium is $\lambda_t = b\,\sigma\,\Gamma_t = b\,\sigma\,(\gamma_o + \gamma' X_t)$. It is function of the p most recent lagged values of the factor x_{t+1}. The recent past (and not only the present value x_t) determine the risk premium level in t.

4.2.4 The Affine Term Structure of Interest Rates

□ The price at date t of the zero-coupon bond with time to maturity h is :

$$B(t, t+h) = \exp(c_h' X_t + d_h)\,, \;\; h \geq 1\,,$$

□ where c_h and d_h satisfies the recursive equations :

$$\begin{aligned} c_h &= -\alpha + \Phi' c_{h-1} + c_{1,h-1}\sigma\gamma = -\alpha + \Phi^{*'} c_{h-1}\,, \\ d_h &= -\beta + c_{1,h-1}(\nu + \gamma_o \sigma) + \tfrac{1}{2} c^2_{1,h-1}\sigma^2 + d_{h-1}\,, \end{aligned}$$

□ with :

$$\Phi^* = \begin{bmatrix} \varphi_1 + \sigma\gamma_1 & \dots & \dots & \varphi_{p-1} + \sigma\gamma_{p-1} & \varphi_p + \sigma\gamma_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix}$$

□ The initial conditions are $c_0 = 0, d_0 = 0$ (or $c_1 = -\alpha, d_1 = -\beta$); $c_{1,h}$ is the first component of the p-dimensional vector c_h [Proof : exercise].

□ The continuously compounded term structure of interest rates is given by:

$$R(t, t+h) = -\frac{1}{h} \log B(t, t+h) = -\frac{c_h'}{h} X_t - \frac{d_h}{h}, \quad h \geq 1,$$

□ For a given date t, any yield $R(t, t+h)$ is an affine function of the factor X_t, that is of the p most recent lagged values of x_{t+1}.

4.2.5 Excess Returns of Zero-Coupon Bonds

□ Under no-arbitrage, and for a fixed maturity T, the one-period geometric zero-coupon bond return process $\rho = [\,\rho(t,T), 0 \leq t \leq T\,]$, where $\rho(t+1,T) = \log[B(t+1,T)] - \log[B(t,T)]$, is given by:

$$\rho(t+1,T) \;=\; r_t - \tfrac{1}{2}\,\omega(t+1,T)^2 + \omega(t+1,T)\Gamma_t - \omega(t+1,T)\,\varepsilon_{t+1}\,,$$

where $\omega(t+1,T) = -(\sigma c_{1,T-t-1})$ [Proof : exercise].

□ This means that the process ρ is such that:

$$\rho(t+1,T)\,|\,\underline{x_t} \;\sim\; N\left[\mu(t+1,T), \omega(t+1,T)^2\right]$$

where $\mu(t+1,T) = r_t - \tfrac{1}{2}\,\omega(t+1,T)^2 + \omega(t+1,T)\Gamma_t\,,$

and $\omega(t+1,T)^2 = (\sigma c_{1,T-t-1})^2\,.$

□ The associated risk premium between t and $t+1$, denoted by $\lambda_t(T)$, is:

$$\lambda_t(T) = \log E_t \exp[\,\rho(t+1,\,T)] - r_t = \omega(t+1,\,T)\,\Gamma_t = \omega(t+1,\,T)(\gamma_o + \gamma' X_t)\,.$$

□ We note that $\Gamma_t = (\gamma_o + \gamma' X_t)$ plays for any T the role of a risk premium per unit of "risk" $\omega(t+1,\,T)$.

□ In particular, the variability of $\lambda_t(T)$ is driven, for a fixed γ different from zero, by the p most recent lagged values of x_{t+1}.

4.2.6 Risk-Neutral Dynamics

□ The risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$\begin{aligned} E_t^{\mathbb{Q}}[\exp(ux_{t+1})] &= \exp\left[u(\nu + \varphi' X_t) - \tfrac{1}{2}(\gamma_o + \gamma' X_t)^2\right] E_t[\exp(\gamma_o + \gamma' X_t + u\sigma)\varepsilon_{t+1}] \\ &= \exp\left[u[(\nu + \sigma\gamma_o) + (\varphi + \sigma\gamma)' X_t] + \tfrac{1}{2}u^2\sigma^2\right], \end{aligned}$$

where $\varphi = [\varphi_1, \ldots, \varphi_p]'$ [Proof : exercise]. Therefore, we get the following result.

□ Under the risk-neutral probability $\mathbb{Q}$, x_{t+1} is an AR(p) process of the following type:

$$x_{t+1} = \nu^* + \varphi_1^* x_t + \ldots + \varphi_p^* x_{t+1-p} + \sigma^* \eta_{t+1},$$

□ with

$$\begin{aligned} \nu^* &= (\nu + \sigma\gamma_o)\,, \;\; \varphi_i^* = (\varphi_i + \sigma\gamma_i) \quad \text{for}\, i \in \{1, \dots, p\} \\ \sigma^* &= \sigma\,, \end{aligned}$$

where $\eta_{t+1} \overset{\mathbb{Q}}{\sim} \mathcal{IIN}(0,1)$. Note that $\varepsilon_{t+1} = \eta_{t+1} + \Gamma_t$.

□ This model can be represented in the following vectorial form :

$$X_{t+1} = \tilde{\nu}^* + \Phi^* X_t + \sigma^* \tilde{\eta}_{t+1}\,,$$

□ where $\tilde{\nu}^* = [\,\nu^*, 0, \dots, 0\,]'$ and $\tilde{\eta}_{t+1} = [\,\eta_{t+1}, 0, \dots, 0\,]'$ are p-dimensional vectors.

4.2.7 The Gaussian AR(p) short-rate model

- We assume that the factor $x_{t+1} = r_{t+1}$ is a Gaussian AR(p) process of the following type:

$$r_{t+1} = \nu + \varphi_1 r_t + \ldots + \varphi_p r_{t-p+1} + \sigma \varepsilon_{t+1} \,,$$

- we have the same SDF, but now the AAO condition $E_t(M_{t,t+1}) = \exp(-r_t)$ implies $\beta = 0$ and $\alpha = (1, 0, \ldots, 0)' \in \mathbb{R}^p$. We have to guarantee that the theoretical formula $R(t, t+h)$ generates, when $h = 1$, exactly the short rate process we have assumed under $\mathbb{P}$.

□ Clearly, under $\mathbb{Q}$ we have:

$$r_{t+1} = \nu^* + \varphi_1^* r_t + \ldots + \varphi_p^* r_{t-p+1} + \sigma \eta_{t+1} \,,$$

□ It is the discrete-time "multiple lags" generalization of the continuous-time Vasicek (1977) model.

4.2.8 The S-Forward Dynamics

□ The S-forward dynamics of x_{t+1} has an AR(p) representation of the following type:

$$x_{t+1} = \nu_S + \varphi_1^* x_t + \ldots + \varphi_p^* x_{t+1-p} + \sigma^* \xi_{t+1} \, ,$$

□ with

$$\nu_S = \nu^* - \sigma^* \omega(t+1, S) \, ,$$

□ and where $\xi_{t+1} \sim \mathcal{IIN}(0,1)$ under $\mathbb{Q}_S$ [Proof : exercise]. Observe that $\varepsilon_{t+1} = \xi_{t+1} - \omega(t+1, S) + \Gamma_t$, where $\Gamma_t = \gamma_o + \gamma' X_t$.

☐ In the S-forward framework, the one-period geometric zero-coupon bond return process is described by the relation:

$$\rho(t+1,T) \;=\; -\omega(t+1,T)\,\xi_{t+1} + r_t - \tfrac{1}{2}\,\omega(t+1,T)^2 + \omega(t+1,T)\omega(t+1,S)\,,$$

☐ with a one-period risk premium given by :

$$\lambda_t^{\mathbb{Q}^{(S)}}(T) = \log E_t^{\mathbb{Q}^{(S)}} \exp\left[\rho(t+1,T)\right] - r_t = \omega(t+1,T)\omega(t+1,S)\,,$$

[Proof : exercise].

☐ Consequently, under the T-forward probability, the one-period risk premium per unit of $\omega(t+1,T)$ is given by the $\omega(t+1,T)$ itself.

4.2.9 Yield Curve Shapes

- ☐ Which kind of yield curve shapes are we able to generate thanks to the introduction of lagged factor values ?

- ☐ Compared to the Gaussian AR(1) case, are we able to generate yield curves closer to the observed ones ?

- ☐ Let us consider (from CRSP) a data set on the U. S. term structure of interest rates (treasury zero-coupon bond yields), covering the period from June 1964 to December 1995. We have 379 monthly observations for each of the nine maturities : 1, 3, 6 and 9 months and 1, 2, 3, 4 and 5 years.

Table 1 : Summary Statistics on U. S. Monthly Yields from June 1964 to December 1995. ACF(k) indicates the empirical autocorrelation between yields $R(t,h)$ and $R(t-k,h)$, with h and k expressed on a monthly basis.

Maturity	1-m	3-m	6-m	9-m	1-yr	2-yr	3-yr	4-yr	5-yr
Mean	0.0645	0.0672	0.0694	0.0709	0.0713	0.0734	0.0750	0.0762	0.0769
Std. Dev.	0.0265	0.0271	0.0270	0.0269	0.0260	0.0252	0.0244	0.0240	0.0237
Skewness	1.2111	1.2118	1.1518	1.1013	1.0307	0.9778	0.9615	0.9263	0.8791
Kurtosis	4.5902	4.5237	4.3147	4.1605	3.9098	3.6612	3.5897	3.5063	3.3531
Minimum	0.0265	0.0277	0.0287	0.0299	0.0311	0.0366	0.0387	0.0397	0.0398
Maximum	0.1640	0.1612	0.1655	0.1644	0.1581	0.1564	0.1556	0.1582	0.1500
ACF(1)	0.9587	0.9731	0.9747	0.9745	0.9727	0.9780	0.9797	0.9802	0.9822
ACF(5)	0.8288	0.8531	0.8579	0.8588	0.8604	0.8783	0.8915	0.8986	0.9053
ACF(10)	0.7278	0.7590	0.7691	0.7699	0.7683	0.7885	0.8021	0.8075	0.8212
ACF(20)	0.4303	0.4631	0.4880	0.4996	0.5156	0.5742	0.6051	0.6193	0.6431
ACF(30)	0.2548	0.2682	0.3016	0.3213	0.3518	0.4358	0.4725	0.4994	0.5187
ACF(40)	0.1362	0.1415	0.1677	0.1853	0.2160	0.3056	0.3427	0.3780	0.3961

☐ The term structure of ZCB yields is, on average:

- upward sloping
- and the yields with larger standard deviation, positive skewness and kurtosis are those with shorter maturities.
- Moreover, yields are **highly autocorrelated** with a persistence which is increasing with the time to maturity.

☐ Let us take as factor the 1-month yield : $r_t = R(t, t + 1\,month)$

☐ **Figures A, B, C and D:** examples of observed yield curves in the data base.

☐ **Figures from 1 to 4:** yield curves generated by a Gaussian AR(1) ATSM.

$\hookrightarrow$ Shapes **can be only** monotone increasing/decreasing, flat or with hump.

☐ **Figures from 5 to 8:** yield curves generated by a Gaussian AR(2) ATSM.

$\hookrightarrow$ Richer but not really realistic shapes.

☐ **Figures from 9 to 12:** yield curves generated by a Gaussian AR(3) ATSM.

$\hookrightarrow$ Richer and more realistic shapes (two humps).

FIGURE A – Observed Yield Curves :
Humpshaped (dashed), Flat (dotted);
Monotone Decreasing (dots and dashes), Monotone Increasing (solid);

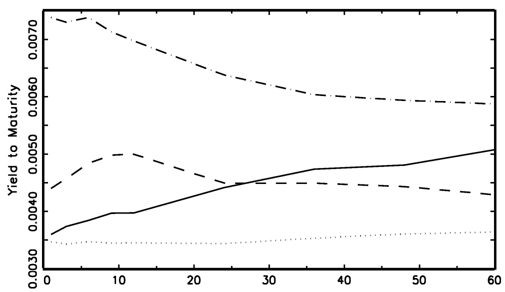


FIGURE B – Observed Yield Curves : J-SHAPES
J-shaped (dashed; dots and dashes), J-Humpshaped (solid);

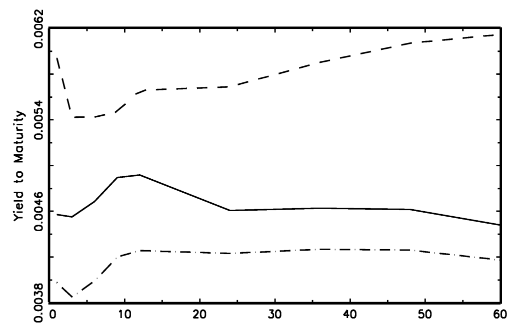


FIGURE C – Observed Yield Curves : L-SHAPES
L-shaped (dashed), Inverted L-shaped (solid);

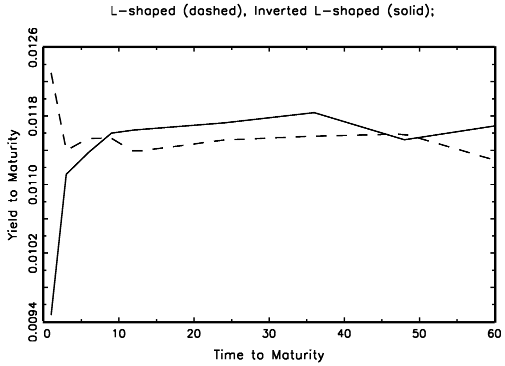


FIGURE D – Observed Yield Curves : L-SHAPES
Inverted L-shaped (dashed and solid);

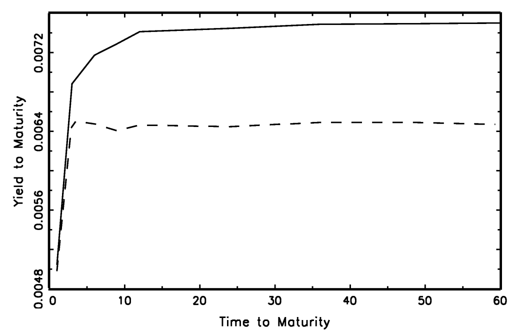


FIGURE 1 – Gaussian AR(1) model;
phi_star = 0.99, r = 0.003; sigma^2 = 0.00000039;
nu_star = 0.00005 (bottom curve) to 0.00030 (top curve);

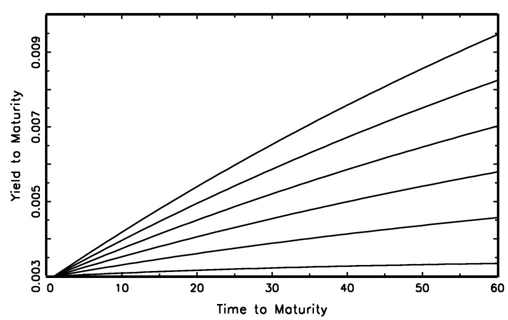


FIGURE 2 – Gaussian AR(1) model;
phi_star = 0.99, r = 0.003; sigma^2 = 0.000008;
nu_star = 0.00010 (bottom curve) to 0.00015 (top curve);

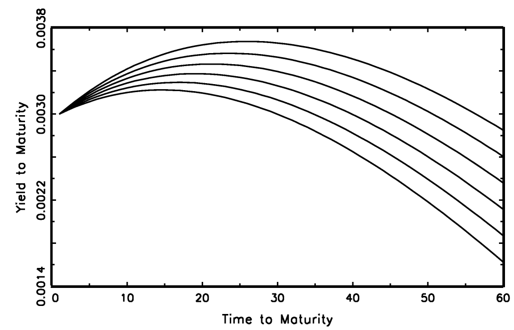


FIGURE 3 – Gaussian AR(1) model;
phi = 0.95; phi_star = 0.87 (bottom curve) to 0.99 (top curve);
r = 0.003; nu_star = 0.00007; sigma^2 = 0.00000039;

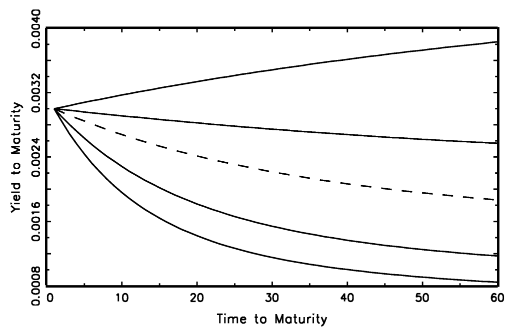


FIGURE 4 – Gaussian AR(1) model;
phi_star = 0.99, r = 0.003; nu_star = 0.00007;
sigma^2 = 0.0000004 (top curve) to 0.0000024 (bottom curve);

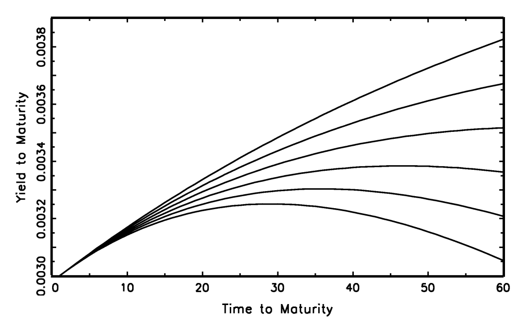


FIGURE 5 – Gaussian AR(2);
phi1_star = 0.74, phi2_star = 0.25;
r_{t+1} = 0.0036, r_{t} = 0.0030; sigma^2 = 0.00000039;
nu_star = 0.00005 (bottom curve) to 0.00030 (top curve)

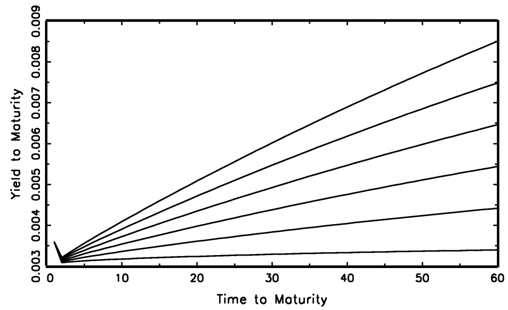


FIGURE 6 – Gaussian AR(2);
phi1_star = 0.74, nu_star = 0.00008;
sigma^2 = 0.00000039, r_{t+1} = 0.0030, r_{t} = 0.0036;
phi2_star = 0.04 (bottom curve) to 0.24 (top curve);

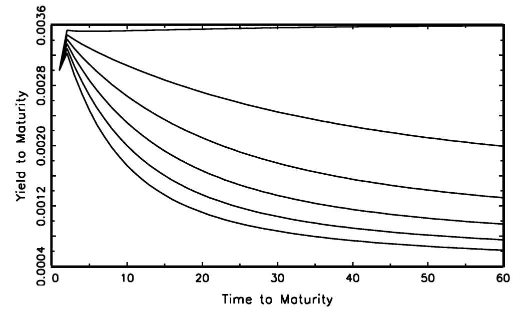


FIGURE 7 – Gaussian AR(2);
phi1_star = 0.74, phi2_star = 0.25;
r_{t+1} = 0.0032, r_{t} = 0.0036; sigma^2 = 0.000008;
nu_star = 0.00010 (bottom curve) to 0.00015 (top curve);

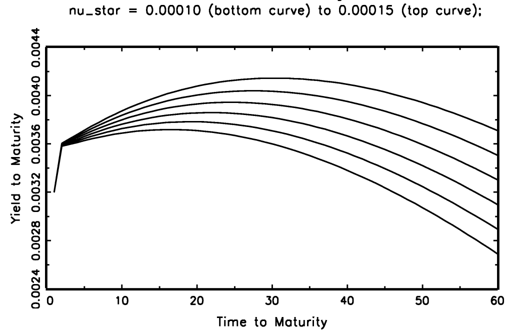


FIGURE 8 – Gaussian AR(2);
phi1_star = 0.74, phi2_star = 0.25;
r_{t+1} = 0.0036, r_{t} = 0.0032; nu_star = 0.00007;
sigma^2 = 0.0000004 (top curve) to 0.0000024 (bottom curve);

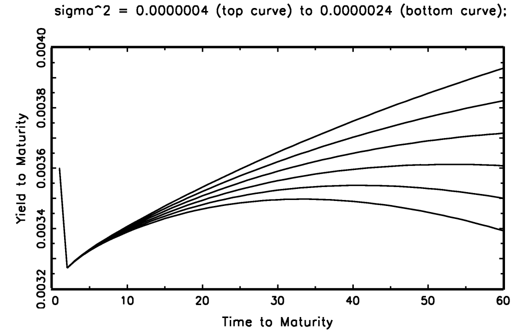


FIGURE 9 – AR(3);
nu^* = 0.00008, sigma^2 = 0.00000039;
phi1^* = 0.79 (bottom) to 0.865 (top);
phi2^* = 0.12 (bottom) to 0.1075 (top);
phi3^* = 0.07 (bottom) to 0.02 (top)

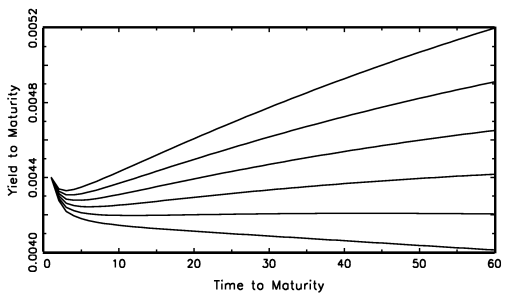


FIGURE 10 – AR(3);
phi1^* = 0.79, phi3^* = 0.07, nu^* = 0.00008;
sigma^2 = 0.00000039; phi2^* = 0.12 (top) to 0.1075 (bottom);

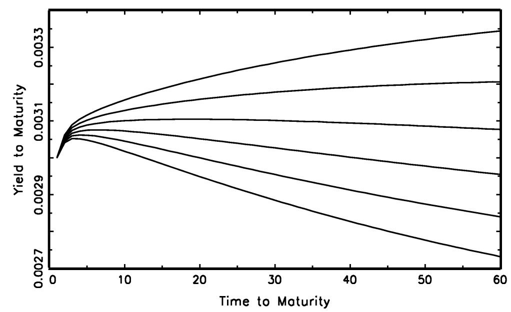


FIGURE 11 – AR(3);
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.05;
sigma^2 = 0.000008; nu^* = 0.00010 (bottom) to 0.00015 (top);

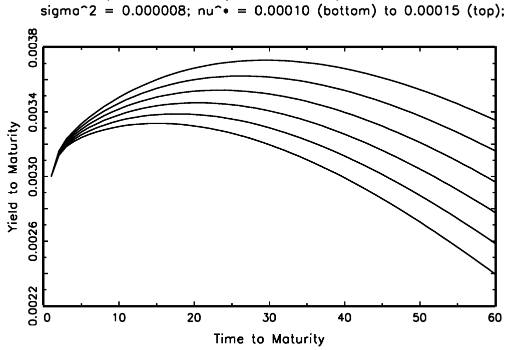
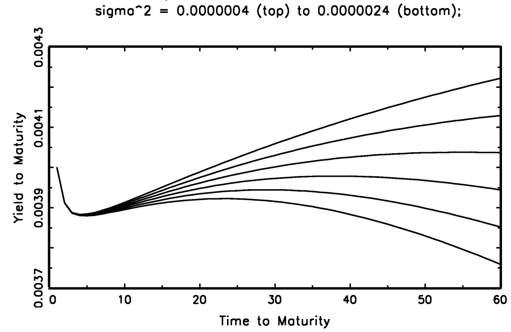


FIGURE 12–Gaussian AR(3);
phi1^* = 0.766, phi2^* = 0.17;
phi3^* = 0.05; nu^* = 0.00008;
sigma^2 = 0.0000004 (top) to 0.0000024 (bottom);



☐ Nevertheless, we have to keep in mind that the shapes we have seen have been generated by **chosen** (*and not estimated* !) **parameter values** !

☐ **If we want to realistically verify** the ability of Gaussian AR(p) ATSMs models to generate yield curves closer to the observed one, we have to:

a) **first**, estimate the parameters of the model

b) **second**, generate the yield curves by means of the yield curve formula, fixing parameter values to their estimated values.

c) **third**, compare them with other possible (competing) yield curve models: which model fit the observed yield curves better (i.e. smallest pricing errors) ?

Using estimated parameters

- □ **Figures from 1 to 4 (slide 30) :** AR(1) model-implied yield curve shapes.
- □ **Figures from 1 to 6 (slide 31):** AR(3) model-implied yield curve shapes.
- □ **Figures from 1 to 6 (slide 32):** AR(4) model-implied yield curve shapes.
- □ **Figures from 1 to 6 (slide 33):** AR(5) model-implied yield curve shapes.
- □ **Figures from 1 to 6 (slide 34):** AR(6) model-implied yield curve shapes.

FIGURE 1 – AR(1);
nu^* = 0.00007036, sigma^2 = 0.00000039;
phi1^* = 0.994368, r_{t+1} = 0.0010;

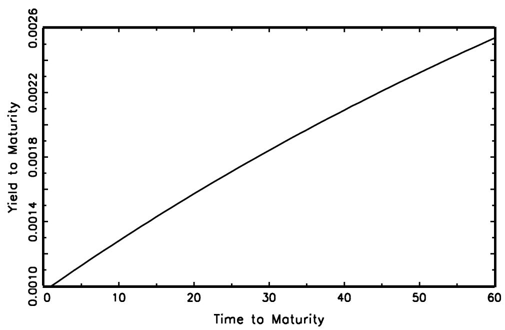


FIGURE 2 – AR(1);
nu^* = 0.00007036, sigma^2 = 0.00000039;
phi1^* = 0.994368, r_{t+1} = 0.0020;

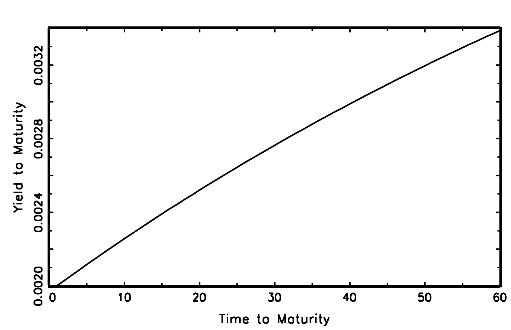


FIGURE 3 – AR(1);
nu^* = 0.00007036, sigma^2 = 0.00000039;
phi1^* = 0.994368, r_{t+1} = 0.0030;

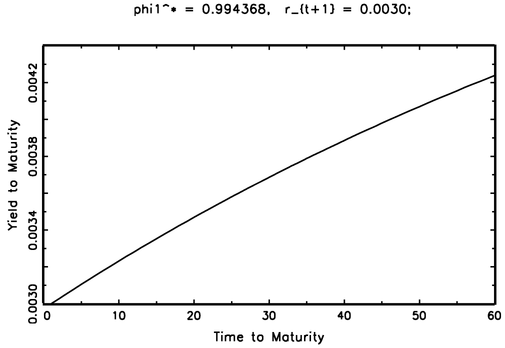


FIGURE 4 – AR(1);
nu^* = 0.00007036, sigma^2 = 0.00000039;
phi1^* = 0.994368, r_{t+1} = 0.0040;

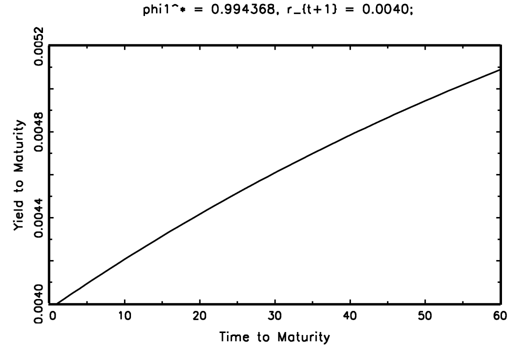


FIGURE 1 - AR(3);
nu^* = 0.000087, sigma^2 = 0.00000039;
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.055;
r_(t-1) = 0.0030, r_(t) = 0.0044, r_(t+1) = 0.0034;

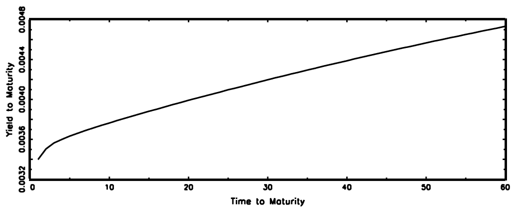


FIGURE 2 - AR(3);
nu^* = 0.000087, sigma^2 = 0.00000039;
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.055;
r_(t-1) = 0.0034, r_(t) = 0.0030, r_(t+1) = 0.0044;

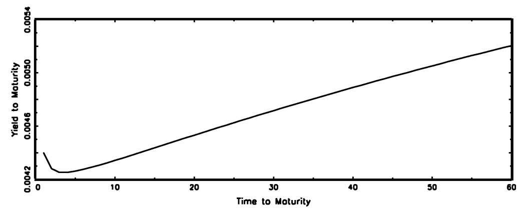


FIGURE 3 - AR(3);
nu^* = 0.000087, sigma^2 = 0.00000039;
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.055;
r_(t-1) = 0.0044, r_(t) = 0.0034, r_(t+1) = 0.0030;

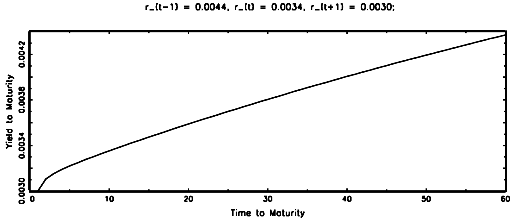


FIGURE 4 - AR(3);
nu^* = 0.000087, sigma^2 = 0.00000039;
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.055;
r_(t-1) = 0.0030, r_(t) = 0.0034, r_(t+1) = 0.0044;

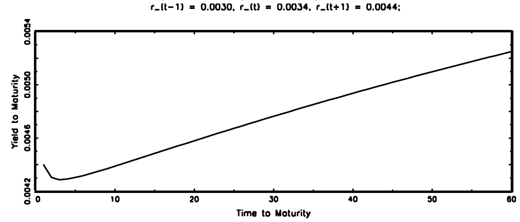


FIGURE 5 - AR(3);
nu^* = 0.000087, sigma^2 = 0.00000039;
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.055;
r_(t-1) = 0.0034, r_(t) = 0.0034, r_(t+1) = 0.0034;

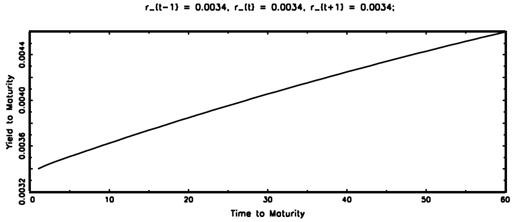


FIGURE 6 - AR(3);
nu^* = 0.000087, sigma^2 = 0.00000039;
phi1^* = 0.766, phi2^* = 0.17, phi3^* = 0.055;
r_(t-1) = 0.0015, r_(t) = 0.0015, r_(t+1) = 0.0015;

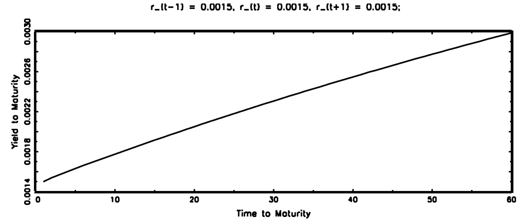


FIGURE 1 – AR(4);
r_(t−2) = 0.0030, r_(t−1) = 0.0034,
r_(t) = 0.0038, r_(t+1) = 0.0044;

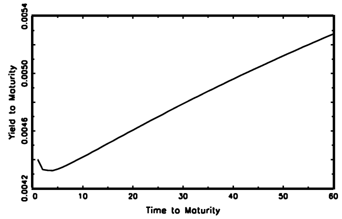


FIGURE 2 – AR(4);
r_(t−2) = 0.0044, r_(t−1) = 0.0038,
r_(t) = 0.0034, r_(t+1) = 0.0030;

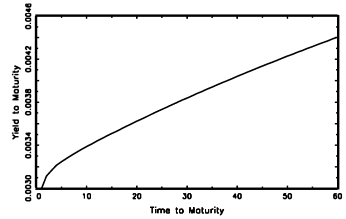


FIGURE 3 – AR(4);
r_(t−2) = 0.0030, r_(t−1) = 0.0038,
r_(t) = 0.0044, r_(t+1) = 0.0034;

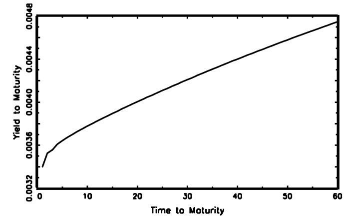


FIGURE 4 – AR(4);
r_(t−2) = 0.0038, r_(t−1) = 0.0030,
r_(t) = 0.0034, r_(t+1) = 0.0044;

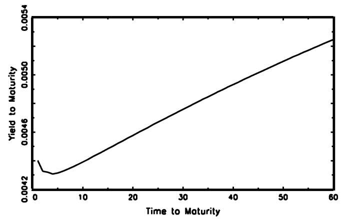


FIGURE 5 – AR(4);
r_(t−2) = 0.0034, r_(t−1) = 0.0038,
r_(t) = 0.0030, r_(t+1) = 0.0044;

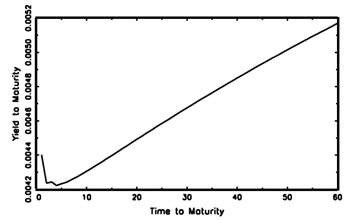


FIGURE 6 – AR(4);
r_(t−2) = 0.0038, r_(t−1) = 0.0034,
r_(t) = 0.0044, r_(t+1) = 0.0030;

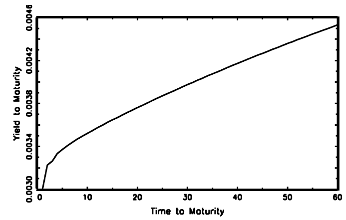


FIGURE 7 – AR(4);
r_(t−2) = 0.0034, r_(t−1) = 0.0034,
r_(t) = 0.0034, r_(t+1) = 0.0034;

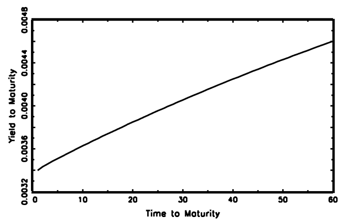


FIGURE 8 – AR(4);
r_(t−2) = 0.0015, r_(t−1) = 0.0015,
r_(t) = 0.0015, r_(t+1) = 0.0015;

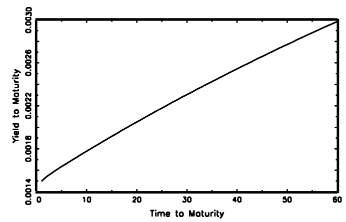


FIGURE 1 – AR(6);
r_{t-4} = 0.0020, r_{t-3} = 0.0030, r_{t-2} = 0.0034,
r_{t-1} = 0.0038, r_{t} = 0.0044, r_{t+1} = 0.0050;

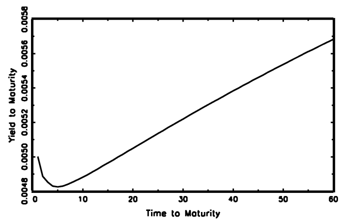


FIGURE 2 – AR(6);
r_{t-4} = 0.0050, r_{t-3} = 0.0044, r_{t-2} = 0.0038,
r_{t-1} = 0.0034, r_{t} = 0.0030, r_{t+1} = 0.0020;

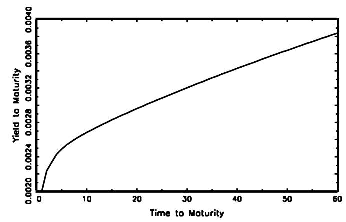


FIGURE 3 – AR(6);
r_{t-4} = 0.0030, r_{t-3} = 0.0038, r_{t-2} = 0.0044,
r_{t-1} = 0.0050, r_{t} = 0.0034, r_{t+1} = 0.0020;

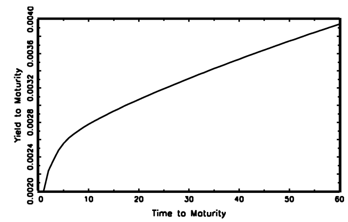


FIGURE 4 – AR(6);
r_{t-4} = 0.0044, r_{t-3} = 0.0038, r_{t-2} = 0.0030,
r_{t-1} = 0.0020, r_{t} = 0.0034, r_{t+1} = 0.0050;

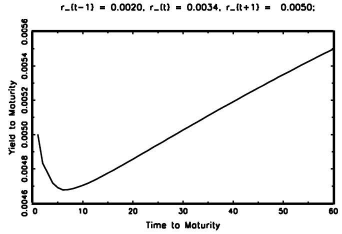


FIGURE 5 – AR(6);
r_{t-4} = 0.0020, r_{t-3} = 0.0034, r_{t-2} = 0.0044,
r_{t-1} = 0.0038, r_{t} = 0.0030, r_{t+1} = 0.0050;

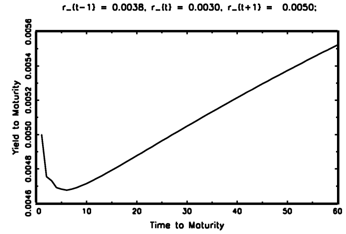


FIGURE 6 – AR(6);
r_{t-4} = 0.0050, r_{t-3} = 0.0038, r_{t-2} = 0.0034,
r_{t-1} = 0.0044, r_{t} = 0.0030, r_{t+1} = 0.0020;

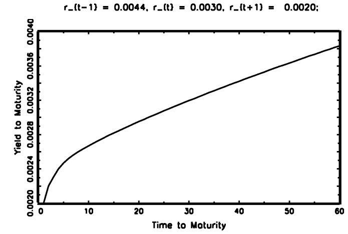


FIGURE 7 – AR(6);
r_{t-4} = 0.0034, r_{t-3} = 0.0044, r_{t-2} = 0.0038,
r_{t-1} = 0.0050, r_{t} = 0.0030, r_{t+1} = 0.0020;

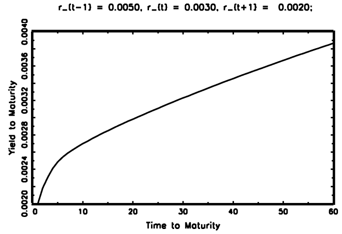


FIGURE 8 – AR(6);
r_{t-4} = 0.0038, r_{t-3} = 0.0020, r_{t-2} = 0.0044,
r_{t-1} = 0.0030, r_{t} = 0.0034, r_{t+1} = 0.0050;

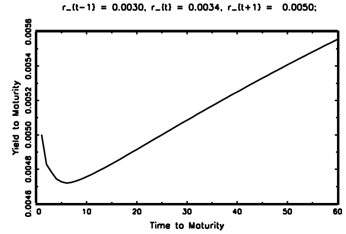
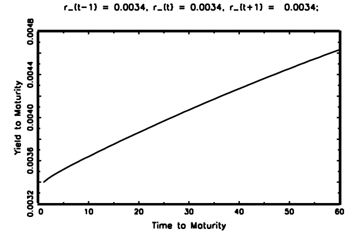


FIGURE 9 – AR(6);
r_{t-4} = 0.0034, r_{t-3} = 0.0034, r_{t-2} = 0.0034,
r_{t-1} = 0.0034, r_{t} = 0.0034, r_{t+1} = 0.0034;



Fixed Income and Credit Risk

Lecture 4 - Part II

Discrete-Time Bivariate Gaussian

VAR(1) Term Structure Models

Outline of Lecture 4 - Part II

4.3 Bivariate Gaussian VAR(1) Factor-Based Term Structure Models

4.3.1 Historical Dynamics

4.3.2 The Stochastic Discount Factor

4.3.3 The Risk Premium

4.3.4 The Affine Term Structure of Interest Rates

4.3.5 Excess Returns of Zero-Coupon Bonds

4.3.6 The Bivariate Gaussian VAR(1) Observable Factor-Based Model

4.3 Gaussian VAR(1) Factor-Based Term Structure Models

4.3.1 Historical Dynamics

- □ We consider our discrete-time economy between dates 0 and T.

- □ x_t is our **factor or a state vector**, and it may be observable, partially observable or unobservable by the econometrician.

- □ The Gaussian AR(p) ATSM is (at estimated parameters) not able to completely explain the variability over time and maturities of the observed yield curves $\Rightarrow$ we need more information, i.e. more factors!

- □ The size of $x_t = (x_{1,t}, x_{2,t})'$ is now assumed to be $K = 2$.

☐ The **historical dynamics** of x_t is defined by the joint distribution of $\underline{x}_T = (x_0, \ldots, x_T)$, denoted by $\mathbb{P}$, or by the conditional probability density function (p.d.f.):

$$f_t(x_{1,t+1}, x_{2,t+1} | \underline{x}_t)\,,$$

☐ or by the **conditional Laplace transform** (L.T.):

$$\varphi_t(u|\underline{x}_t) = E[\exp(u_1\, x_{1,t+1} + u_2\, x_{2,t+1}) \,|\, \underline{x}_t] = E[\exp(u' x_{t+1}) | \underline{x}_t] = E_t[\exp(u' x_{t+1})]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^2$ (containing zero).

☐ We also introduce the **conditional Log-Laplace transform**:

$$\psi_t(u|\underline{x}_t) = \psi_t(u) = \mathrm{Log}[\varphi_t(u|\underline{x}_t)]\,.$$

□ Let us assume that the (non observable) 2-dimensional factor $x_{t+1} = (x_{1,t+1}, x_{2,t+1})'$ is a Gaussian VAR(1) process of the following type:

$$x_{t+1} \;=\; \nu + \Phi x_t + \Sigma \varepsilon_{t+1} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} + \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} x_t + \Sigma \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

where ε_t is a 2-dimensional Gaussian white noise with $\mathcal{N}(0, I_2)$ distribution.

□ $E_t[x_{t+1}] = \nu + \Phi x_t$ and $V_t[x_{t+1}] = \Sigma\Sigma' = \Omega$ (symmetric positive semi-definite),

$\Rightarrow$ $x_{t+1} \,|\, x_t \sim N_K(\nu + \Phi x_t, \Omega)$ (under $\mathbb{P}$).

□ At date t, the k-step ahead forecast (denoted $x^e_{t+k\,|\,t}$) with a VAR(1) model:

$$x^e_{t+k\,|\,t} := E_t[x_{t+k}] = (I_2 + \Phi + \ldots + \Phi^{k-1})\,\nu + \Phi^k\, x_t\,.$$

☐ We do not have a unique decomposition of Ω:

- $\Sigma = (\sigma_{i,j})$ can be chosen lower triangular (in general : Choleski decomposition) to guarantee $\Omega > 0$ and symmetric.

- Using Choleski ($\Sigma = (\sigma^c_{i,j})$) we impose $\sigma^c_{i,i} > 0$, $i \in \{1, 2\}$, to solve identification problems.

☐ Under stationarity (i.e. all values of z such that $|\, I_2 - \Phi z \,| = 0$ lie outside the unit circle), we have

- $E[x_t] = (I_2 - \Phi)^{-1}\, \nu$ and $V[x_t]$ is such that $vec(V[x_t]) = (I_{2^2} - \Phi \otimes \Phi)^{-1}\, vec(\Omega)$,

$\Rightarrow\ x_t \sim N_2\,(E[x_t], V[x_t])$ (under $\mathbb{P}$).

☐ Let us remember that the Laplace transform of a 2-dimensional Gaussian random variable $Y \sim N_2(\mu, \Xi)$, with $\mu = (\mu_1, \mu_2)'$, $\Xi_{11} = V[Y_1]$, $\Xi_{22} = V[Y_2]$, $\Xi_{12} = Cov[Y_1, Y_2] = \Xi_{21}$, is:

$$\begin{aligned} \varphi(u) &= E[\exp(u_1 Y_1 + u_2 Y_2)] = \exp\left(u'\mu + \frac{1}{2}\, u'\Xi u\right) \\ &= \exp\left[(u_1\mu_1 + u_2\mu_2) + \frac{1}{2}\,(u_1^2\, V[Y_1] + u_2^2\, V[Y_2] + 2\, u_1\, u_2 Cov[Y_1, Y_2])\right] \end{aligned}$$

☐ This means that:

$$\varphi_t(u|\underline{x}_t) = \varphi_t(u) = \exp\left[u'(\nu + \Phi x_t) + \frac{1}{2}\, u'\,\Omega u\right] = \exp\left[(u'\,\nu + \frac{1}{2}\, u'\,\Omega u) + u'\,\Phi x_t\right],$$

and

$$E[\exp(u' x_t)] = \exp\left[u' E[x_t] + \frac{1}{2}\, u'\, V[x_t] u\right].$$

4.3.2 The Stochastic Discount Factor

- We specify the following SDF:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' x_t + \Gamma_t' \varepsilon_{t+1} - \frac{1}{2}\Gamma_t'\Gamma_t\right],$$

- the coefficients $\alpha = [\alpha_1, \alpha_2]'$ and β are path independent,

- $\Gamma_t = \Gamma(X_t) = (\gamma_o + \gamma x_t)$, where $\gamma_o = (\gamma_{1,o}, \gamma_{2,o})'$ and γ is a $(2,2)$-matrix:

$$\begin{aligned}\Gamma_{1,t} &= \gamma_{o,1} + \gamma_{1,1}\,x_{1,t} + \gamma_{1,2}\,x_{2,t} = \gamma_{1,o} + \widetilde{\gamma}_1' x_t \\ \Gamma_{2,t} &= \gamma_{o,2} + \gamma_{2,1}\,x_{1,t} + \gamma_{2,2}\,x_{2,t} = \gamma_{2,o} + \widetilde{\gamma}_2' x_t\,.\end{aligned}$$

□ This means that, at any date t, the risk-correction coefficients associated to the first and second factor, i.e. $\Gamma_{1,t}$ and $\Gamma_{2,t}$ respectively, are a linear combination of BOTH scalar factors $x_{1,t}$ and $x_{2,t}$.

□ The no-arbitrage restriction $E_t(M_{t,t+1}) = \exp(-r_t)$, implies the relation

$r_t = \beta + \alpha' x_t = \beta + \alpha_1 \, x_{1,t} + \alpha_2 \, x_{2,t}$.

→ Thus, we now assume that the short rate has a dynamics explained by two variables (two factors) like, for instance, short and long rate, short rate and spread, one yield and one macro variable, level and slope factors.

4.3.3 The Risk Premium

☐ Given the following definition of risk premium:

$$\lambda_t = \log E_t \left(\frac{P_{t+1}}{P_t} \right) - r_t = \log E_t \exp(y_{t+1}) - r_t \,,$$

☐ and given the payoff $\exp(-b' x_{t+1})$ at $t+1$, its price in t is given by:

$$\begin{aligned} P_t &= E_t \left[M_{t,t+1} P_{t+1} \right] = \exp\left[-r_t - b'(\nu + \Phi x_t) - b' \Sigma \Gamma_t + \frac{1}{2} b' \Omega b \right] , \\ E_t P_{t+1} &= E_t[\exp(-b' x_{t+1})] = \exp\left[-b'(\nu + \Phi' x_t) + \frac{1}{2} b' \Omega b \right] . \end{aligned}$$

☐ the risk premium is $\lambda_t = b' \, \Sigma \, \Gamma_t = b' \, \Sigma \, (\gamma_o + \gamma x_t)$. It is function of the 2-dimensional factor x_t.

4.3.4 The Affine Term Structure of Interest Rates

☐ The price at date t of the zero-coupon bond with time to maturity h is :

$$B(t, t+h) = \exp(C_h' x_t + D_h) = \exp(C_{1,h} x_{1,t} + C_{2,h} x_{2,t} + D_h)\,,\ \ h \geq 1\,,$$

☐ where c_h and d_h satisfies, for $h \geq 1$, the recursive equations:

$$\left\{\begin{array}{lcl} C_h & = & -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} \\ & = & -\alpha + \Phi^{*'} C_{h-1}\,, \\ D_h & = & -\beta + C_{h-1}'(\nu + \Sigma\gamma_o) + \frac{1}{2} C_{h-1}'(\Sigma\Sigma') C_{h-1} + D_{h-1} \\ & = & -\beta + C_{h-1}'\nu^* + \frac{1}{2} C_{h-1}' \Omega C_{h-1} + D_{h-1}\,, \end{array}\right.$$

☐ with initial conditions $C_0 = 0, D_0 = 0$ (or $C_1 = -\alpha, D_1 = -\beta$).

□ The affine term structure of interest rates formula is:

$$\begin{aligned} R(t, t+h) = -\frac{1}{h} \log B(t, t+h) &= -\frac{C_h'}{h} x_t - \frac{D_h}{h} \\ &= -\frac{1}{h} \left(C_{1,h} x_{1,t} + C_{2,h} x_{2,t} + D_h \right), \quad h \geq 1, \end{aligned}$$

□ For a given date t, any yield $R(t, t+h)$ is an affine function of the 2-dimensional factor $x_t = (x_{1,t}, x_{2,t})'$.

□ This is the discrete-time equivalent of the bivariate (continuous-time affine) Vasicek model.

4.3.5 Gaussian Bivariate VAR(1) Observable Factor-Based Model

☐ The 2-dimensional factor (x_t) can be considered as a vector of two yields: the first component is assumed to be the short rate r_t and the second one is the long rate R_t.

☐ More precisely, we assume:

$$x_t = \begin{bmatrix} R(t, t+1) \\ R(t, t+H) \end{bmatrix}$$

where $R(t, t+1) = r_t$ and $R(t, t+H) = R_t$.

☐ we can start better understanding the role of the no-arbitrage restrictions.

□ First, I have to impose that $R(t, t+1) = r_t$. This condition generates the AAO restriction:

$$R(t, t+1) = \beta + \alpha' x_t = \beta + \alpha_1 r_t + \alpha_2 R_t = r_t$$

$$\Leftrightarrow \beta = 0\,,\;\; \alpha_1 = 1\,,\;\; \alpha_2 = 0\,,$$

These conditions are equivalent to $C_1 = -(1, 0)$ and $D_1 = 0$.

□ Second, I have to impose that $R(t, t+H) = R_t$ for any t. In this case we have:

$$-\frac{1}{H}[C_{1,H}\, r_t + C_{2,H}\, R_t + D_H] = R_t$$

$$\Leftrightarrow C_{1,H}\, r_t + C_{2,H}\, R_t + D_H = -H R_t$$

$$\Leftrightarrow C_{1,H} = 0\,,\;\; C_{2,H} = -H\,,\;\; D_H = 0\,,$$

that is $C_H = -H\,(0, 1)'$ and $D_H = 0$.

☐ In this case, the absence of arbitrage conditions for the 2 yields in x_t imply :

$$\begin{aligned}(i)\; C_1 &= -(1,0)', \quad D_1 = 0\,,\\ (ii)\; C_H &= -H\,(0,1)', \quad D_H = 0\,.\end{aligned}$$

☐ The first set of conditions is used as initial value in the recursive equations (C_h, D_h).

☐ The second condition imply restrictions on model parameters which must be taken into account at the estimation stage. We have to impose to the yield-to-maturity formula to pass through the yields in x_t.

Fixed Income and Credit Risk

Lecture 4 - Part III

Discrete-Time Multivariate Gaussian

VAR(p) Term Structure Models

Outline of Lecture 4 - Part III

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4.4 Gaussian VAR(1) Factor-Based Term Structure Models

4.4.1 Historical Dynamics

- ☐ We consider our discrete-time economy between dates 0 and T.

- ☐ x_t is our **factor or a state vector**, and it may be observable, partially observable or unobservable by the econometrician.

- ☐ The Gaussian AR(p) ATSM is (at estimated parameters) not able to completely explain the variability over time and maturities of the observed yield curves $\Rightarrow$ we need more information, i.e. more factors!

- ☐ The size of x_t is now assumed to be $K > 1$.

☐ The **historical dynamics** of x_t is defined by the joint distribution of $\underline{x}_T = (x_0, \ldots, x_T)$, denoted by $\mathbb{P}$, or by the conditional probability density function (p.d.f.):

$$f_t(x_{t+1}|\underline{x}_t)\,,$$

☐ or by the **conditional Laplace transform** (L.T.):

$$\varphi_t(u|\underline{x}_t) = \varphi_t(u) = E[\exp(u'x_{t+1})|\underline{x}_t] = E_t[\exp(u'x_{t+1})]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^K$ (containing zero).

☐ We also introduce the **conditional Log-Laplace transform**:

$$\psi_t(u|\underline{x}_t) = \psi_t(u) = \text{Log}[\varphi_t(u|\underline{x}_t)]\,.$$

☐ Let us assume that the (non observable) K-dimensional factor $x_{t+1} = (x_{1,t+1}, \ldots, x_{K,t+1})'$ is a Gaussian VAR(1) process of the following type:

$$x_{t+1} = \nu + \Phi x_t + \Sigma \varepsilon_{t+1},$$

where $\varepsilon_{t+1} = (\varepsilon_{1,t+1}, \ldots, \varepsilon_{K,t+1})$ is a K-dimensional Gaussian white noise with $\mathcal{N}(0, I_K)$ distribution.

☐ $E_t[x_{t+1}] = \nu + \Phi x_t$ and $V_t[x_{t+1}] = \Sigma\Sigma' = \Omega$ (symmetric positive semi-definite),

$\Rightarrow x_{t+1} \,|\, x_t \sim N_K(\nu + \Phi x_t, \Omega)$ (under $\mathbb{P}$).

☐ We do not have a unique decomposition of Ω:

- $\Sigma = (\sigma_{i,j})$ can be chosen lower triangular (in general : Choleski decomposition) to guarantee $\Omega > 0$ and symmetric.

- Using Choleski ($\Sigma = (\sigma^c_{i,j})$) we impose $\sigma^c_{i,i} > 0$, $i \in \{1, \dots, K\}$, to solve identification problems.

☐ Under stationarity (i.e. all values of z such that $|\, I_K - \Phi z \,| = 0$ lie outside the unit circle), we have

- $E[x_t] = (I_K - \Phi)^{-1}\,\nu$ and $V[x_t]$ is such that $vec(V[x_t]) = (I_{K^2} - \Phi \otimes \Phi)^{-1}\,vec(\Omega)$,

$\Rightarrow$ $x_t \sim N_K\,(E[x_t], V[x_t])$ (under $\mathbb{P}$).

- Let us remember that the Laplace transform of a K-dimensional Gaussian random variable $Y \sim N_K(\mu, \Xi)$ is:

$$\varphi(u) = E[\exp(u'Y)] = \exp\left(u'\mu + \frac{1}{2}\,u'\Xi u\right) .$$

- This means that:

$$\varphi_t(u|\underline{x}_t) = \varphi_t(u) = \exp\left[u'(\nu + \Phi x_t) + \frac{1}{2}\,u'\,\Omega u\right] ,$$

- and

$$E[\exp(u'x_t)] = \exp\left[u'E[x_t] + \frac{1}{2}\,u'\,V[x_t]u\right] .$$

4.4.2 The Stochastic Discount Factor

☐ We specify the following SDF:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' x_t + \Gamma_t' \varepsilon_{t+1} - \frac{1}{2}\Gamma_t'\Gamma_t\right] ,$$

☐ the coefficients $\alpha = [\alpha_1, \ldots, \alpha_K]'$ and β are path independent,

☐ $\Gamma_t = \Gamma(X_t) = (\gamma_o + \gamma x_t)$, where $\gamma_o = (\gamma_{1,o}, \ldots, \gamma_{K,o})'$ and γ is a (K, K)-matrix:

$$\begin{aligned} \Gamma_{1,t} &= \gamma_{1,o} + \gamma_{1,1}\, x_{1,t} + \gamma_{1,2}\, x_{2,t} + \ldots + \gamma_{1,K}\, x_{K,t} = \gamma_{1,o} + \widetilde{\gamma}_1' x_t \\ &\vdots \\ \Gamma_{K,t} &= \gamma_{K,o} + \gamma_{K,1}\, x_{1,t} + \gamma_{K,2}\, x_{2,t} + \ldots + \gamma_{K,K}\, x_{K,t} = \gamma_{K,o} + \widetilde{\gamma}_K' x_t\,. \end{aligned}$$

□ This means that, at any date t, the risk-correction coefficient associated to the j^{th} factor, i.e. $\Gamma_{j,t}$, is a linear combination of ALL the K scalar factors.

□ The no-arbitrage restriction $E_t(M_{t,t+1}) = \exp(-r_t)$, implies the relation $r_t = \beta + \alpha' x_t = \beta + \alpha_1 x_{1,t} + \ldots + \alpha_K x_{K,t}$.

→ Now, the short rate is explained by a linear combination of K variables that we can select as a mix of yields, latent factors (level/slope) and macro variables.

4.4.3 The Risk Premium

- Given the risk premium :

$$\lambda_t = \log E_t \left(\frac{P_{t+1}}{P_t} \right) - r_t = \log E_t \exp(y_{t+1}) - r_t \, ,$$

- and given the payoff $\exp(-b' x_{t+1})$ at $t+1$, its price in t is given by:

$$\begin{aligned} P_t &= E_t \left[M_{t,t+1} P_{t+1} \right] = \exp \left[-r_t - b'(\nu + \Phi x_t) - b' \Sigma \Gamma_t + \frac{1}{2} b' \Omega b \right] , \\ E_t P_{t+1} &= E_t [\exp(-b' x_{t+1})] = \exp \left[-b'(\nu + \Phi' x_t) + \frac{1}{2} b' \Omega b \right] . \end{aligned}$$

- the risk premium is $\lambda_t = b' \, \Sigma \, \Gamma_t = b' \, \Sigma \, (\gamma_o + \gamma x_t)$. It is function of the K-dimensional factor x_t.

4.4.4 The Affine Term Structure of Interest Rates

☐ The price at date t of the zero-coupon bond with time to maturity h is :

$$B(t, t+h) = \exp(C_h' x_t + D_h) = \exp(C_{1,h} x_{1,t} + \ldots + C_{K,h} x_{K,t} + D_h)\,, \;\; h \geq 1\,,$$

☐ where c_h and d_h satisfies, for $h \geq 1$, the recursive equations:

$$\left\{ \begin{array}{rcl} C_h & = & -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} \\ & = & -\alpha + \Phi^{*'} C_{h-1}\,, \\ D_h & = & -\beta + C_{h-1}'(\nu + \Sigma\gamma_o) + \frac{1}{2} C_{h-1}'(\Sigma\Sigma') C_{h-1} + D_{h-1} \\ & = & -\beta + C_{h-1}'\nu^* + \frac{1}{2} C_{h-1}' \Omega C_{h-1} + D_{h-1}\,, \end{array} \right.$$

☐ with initial conditions $C_0 = 0, D_0 = 0$ (or $C_1 = -\alpha, D_1 = -\beta$).

- The (continuously compounded) affine term structure of interest rates is given by:

$$R(t, t+h) = -\frac{1}{h} \log B(t, t+h) = -\frac{C_h'}{h} x_t - \frac{D_h}{h}, \quad h \geq 1,$$

- For a given date t, any yield $R(t, t+h)$ is an affine function of the K-dimensional factor $x_t = (x_{1,t}, \ldots, x_{K,t})'$.

- This is the discrete-time equivalent of the multivariate (continuous-time affine) Vasicek model.

4.4.5 Excess Returns of Zero-Coupon Bonds

- Under no-arbitrage, and for a fixed maturity T, the one-period geometric zero-coupon bond return process $\rho = [\,\rho(t,T), 0 \leq t \leq T\,]$, where $\rho(t+1,T) = \log[B(t+1,T)] - \log[B(t,T)]$, is given by:

$$\rho(t+1,T) \;=\; r_t - \tfrac{1}{2}\omega(t+1,T)'\omega(t+1,T) + \omega(t+1,T)'\Gamma_t - \omega(t+1,T)'\,\varepsilon_{t+1}\,,$$

where $\omega(t+1,T) = -(\Sigma' C_{T-t-1})$ is an K-dimensional vector.

□ The associated risk premium, between t and $t+1$, is given by :

$$\lambda_t(T) \;=\; \omega(t+1,\,T)'\,\Gamma_t = \sum_{i=1}^{K} \omega_i(t+1,\,T)\,\Gamma_{i,t}\,,$$

where $\omega(t+1,\,T) = [\omega_1(t+1,\,T), \ldots, \omega_K(t+1,\,T)]'$.

□ It is important to highlight that, in this multivariate setting, the magnitude of $\lambda_t(T)$ is given by a linear combination of the K scalar risk premia $\Gamma_{i,t} = \gamma_{o,i} + \widetilde{\gamma}_i' x_t$.

□ In other words, ALL scalar factors $x_{i,t}$, with $i \in \{1, \ldots, K\}$, determine the magnitude and the variability over time of ANY (scalar) risk premia $\Gamma_{i,t}$.

4.4.6 Risk-Neutral Dynamics

- The risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$
\begin{aligned}
E_t^{\mathbb{Q}}[\exp(u'x_{t+1})] &= E_t\left[\frac{M_{t,t+1}}{E_t(M_{t,t+1})}\exp(u'x_{t+1})\right] \\
&= E_t\left[\exp\left(\Gamma_t'\,\varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma_t'\Gamma_t + u'x_{t+1}\right)\right] \\
&= \exp\left[u'(\nu + \Phi x_t) - \tfrac{1}{2}\Gamma_t'\Gamma_t)\right] E_t[\exp(\Gamma_t + \Sigma' u)'\varepsilon_{t+1}] \\
&= \exp\left[u'[(\nu + \Sigma\gamma_o) + (\Phi + \Sigma\gamma)x_t] + \tfrac{1}{2}u'(\Sigma\Sigma')u\right] .
\end{aligned}
$$

- Under the risk neutral probability $\mathbb{Q}$, x_{t+1} is an K-dimensional VAR(1) process of the following type:

$$x_{t+1} = \nu^* + \Phi^* x_t + \Sigma^* \eta_{t+1}\,,$$

□ with

$$\nu^* \;=\; (\nu + \sigma\gamma_o)\,,\;\; \Phi^* = (\Phi + \Sigma\gamma)\,,\;\; \Sigma^* = \Sigma\,,$$

□ and where η_{t+1} is (under $\mathbb{Q}$) an K-dimensional Gaussian white noise with $\mathcal{N}(0, I_K)$ distribution.

□ In the risk-neutral framework, for a fixed maturity T, the one-period geometric zero-coupon bond return process satisfies the relation:

$$\rho(t+1, T) = r_t - \frac{1}{2}\,\omega(t+1, T)'\omega(t+1, T) - \omega(t+1, T)'\,\eta_{t+1}\,,$$

with a risk premium $\lambda_t^{\mathbb{Q}}(T) = 0$.

4.4.7 Gaussian VAR(1) Observable Factor-Based Model

☐ The K-dimensional factor (x_t) can be considered as a vector of yields at different maturities in which the first component is assumed to be the short rate r_t.

☐ More precisely, we assume:

$$x_t = \begin{bmatrix} R(t, t+h_1) \\ R(t, t+h_2) \\ \vdots \\ R(t, t+h_K) \end{bmatrix}$$

where $R(t, t+h_1) = R(t, t+1) = r_t$ and $h_1 < h_2 < \ldots < h_K$.

☐ let us see how no-arbitrage restrictions apply in this general setting.

☐ In this case, the absence of arbitrage conditions for the K yields in x_t imply :

$$\begin{aligned} (i)\; C_1 &= -e_1\,, \quad D_1 = 0\,, \\ (ii)\; C_{h_j} &= -h_j\, e_{h_j}\,, \quad D_{h_j} = 0\,, \forall\, j \in \{2, \ldots, K\}\,, \end{aligned}$$

where e_{h_j} denotes the h_j^{th} element of the canonical basis in $\mathbb{R}^K$.

☐ The first set of conditions is used as initial value in the recursive equations (C_h, D_h).

☐ The second set of $(K-1)$ conditions imply restrictions on model parameters which must be taken into account at the estimation stage. We have to impose to the yield-to-maturity formula to pass through the yields in x_t.

4.4.8 The S-Forward Dynamics

- The S-forward dynamics of the K-dimensional factor x_{t+1} has an VAR(1) representation of the following type:

$$x_{t+1} = \nu_S + \Phi^* x_t + \Sigma^* \xi_{t+1} \,,$$

- with

$$\nu_S = \nu^* - \Sigma^* \omega(t+1, S) \,,$$

and where $\xi_{t+1} \sim \mathcal{IIN}(0, I)$ under $\mathbb{Q}^{(S)}$.

☐ The one-period geometric zero-coupon bond return process is given by:

$$\begin{aligned}\rho(t+1,T) &= r_t - \omega(t+1,T)'\,\xi_{t+1} - \\ &\qquad \tfrac{1}{2}\,\omega(t+1,T)'\omega(t+1,T) + \omega(t+1,T)'\omega(t+1,S)\,,\end{aligned}$$

☐ with one-period risk premium given by :

$$\lambda_t^{\mathbb{Q}^{(S)}}(T) = \log E_t^{\mathbb{Q}^{(S)}} \exp\left[\rho(t+1,T)\right] - r_t = \omega(t+1,T)'\omega(t+1,S)\,.$$

4.5.1 Gaussian VAR(p) Factor-Based Term Structure Models

4.5.1 Historical Dynamics, SDF and Affine Term Structure

□ Let us assume now that the latent factor $x_{t+1} = (x_{1,t+1}, \ldots, x_{K,t+1})'$ driving the term structure is an K-dimensional VAR(p) process of the following type:

$$\begin{aligned} x_{t+1} &= \nu + \Phi_1 x_t + \ldots + \Phi_p x_{t+1-p} + \Sigma \varepsilon_{t+1} \\ &= \nu + \Phi X_t + \Sigma \varepsilon_{t+1}\,, \end{aligned} \tag{1}$$

where ε_{t+1} is a K-dimensional Gaussian white noise with $\mathcal{N}(0, I_K)$ distribution.

□ Σ and Φ_j, for each $j \in \{1, \ldots, p\}$, are (K, K) matrices and Σ can be chosen, for instance, lower triangular (Choleski decomposition).

- $\Phi = [\Phi_1, \ldots, \Phi_p]$ is an (K, Kp) matrix, ν is an K-dimensional vector, while $X_t = (x_t', \ldots, x_{t+1-p}')'$ is an (Kp)-dimensional vector.

- The model can be represented in the following (Kp)-dimensional AR(1) form:

$$X_{t+1} \;=\; \widetilde{\Phi} X_t + [\nu + \Sigma \varepsilon_{t+1}] e_1\,, \tag{2}$$

  where e_1 is a vector of size (Kp), with all entries equal to zero except for the first K elements which are all equal to one

- and where

$$\widetilde{\Phi} = \begin{bmatrix} \Phi_1 & \ldots & \ldots & \Phi_{p-1} & \Phi_p \\ I_K & \mathbf{0}_K & \ldots & \mathbf{0}_K & \mathbf{0}_K \\ \mathbf{0}_K & I_K & \ldots & \mathbf{0}_K & \mathbf{0}_K \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0}_K & \ldots & \ldots & I_K & \mathbf{0}_K \end{bmatrix} \text{ is a } (Kp, Kp) \text{ matrix.}$$

□ $E_t[x_{t+1}] = \nu + \Phi_1 x_t + \ldots + \Phi_p x_{t+1-p}$ and $V_t[x_{t+1}] = \Sigma\Sigma' = \Omega$,

$\Rightarrow\ x_{t+1} \,|\, x_t \sim N(\nu + \Phi X_t, \Omega)$ (under $\mathbb{P}$).

□ Under stationarity (i.e. all values of z such that $|\, I_k - \sum_{j=1}^{p} \Phi_j z^j \,| = 0$ lie outside the unit circle), we have $E[x_t] = \left(I_K - \sum_{j=1}^{p} \Phi_j\right)^{-1} \nu$ and $V[x_t]$ [see Hamilton (1994, Chapter 10) and Lutkepohl (2005, Chapter 2)],

$\Rightarrow\ x_t \sim N\left(E[x_t], V[x_t]\right)$ (under $\mathbb{P}$).

☐ the SDF is defined as :

$$M_{t,t+1} \;=\; \exp\left[-\beta - \alpha' X_t + \Gamma_t' \varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t'\Gamma_t\right] \,, \tag{3}$$

☐ where $\Gamma_t = \gamma_o + \widetilde{\Gamma} X_t$, $\Gamma_t = \left[\,\Gamma_{1,t}, \ldots, \Gamma_{K,t}\right]'$ and with:

$$\Gamma_{i,t} \;=\; \gamma_{o,i} + \sum_{j=1}^{p} \widetilde{\gamma}_{i,j}' x_{t-j+1} \,, \;\; i \in \{1, \ldots, K\} \tag{4}$$

with $\gamma_o = [\gamma_{o,1}, \ldots, \gamma_{o,K}]'$ a K-dimensional vector,

☐ and where

$$\widetilde{\Gamma} = \begin{bmatrix} \widetilde{\gamma}_{1,1}' & \cdots & \cdots & \widetilde{\gamma}_{1,p-1}' & \widetilde{\gamma}_{1,p}' \\ \widetilde{\gamma}_{2,1}' & \cdots & \cdots & \widetilde{\gamma}_{2,p-1}' & \widetilde{\gamma}_{2,p}' \\ \vdots & & \ddots & \vdots & \vdots \\ \widetilde{\gamma}_{K,1}' & \cdots & \cdots & \widetilde{\gamma}_{K,p-1}' & \widetilde{\gamma}_{K,p}' \end{bmatrix} \text{ is a } (K, Kp) \text{ matrix.} \tag{5}$$

☐ Moreover, assuming the absence of arbitrage opportunities for r_t we get $r_t = \beta + \alpha' X_t$, where α is a (Kp)-dimensional vector.

☐ It is also easy to verify that the risk premium, for an asset providing the payoff $\exp(-b' x_{t+1})$ at $t+1$, is $\lambda_t = b'\Sigma\Gamma_t = b'\Sigma(\gamma_o + \widetilde{\Gamma} X_t)$.

☐ This means that the date-t risk-premium λ_t is determined by a linear combination of the p most recent lagged values of the K scalar factors $x_{i,t+1}$ with $i \in \{1, \ldots, K\}$.

□ In the Gaussian VAR(p) Factor-Based Term Structure Model, the price at date t of the zero-coupon bond with time to maturity h is :

$$B(t, t+h) = \exp(C_h' X_t + D_h)\,, \tag{6}$$

□ where C_h and D_h satisfies, for $h \geq 1$, the recursive equations :

$$\left\{ \begin{array}{rcl} C_h & = & -\alpha + \widetilde{\Phi}' C_{h-1} + (\Sigma \widetilde{\Gamma})' C_{1,h-1} \\ & = & -\alpha + \widetilde{\Phi}^{*'} c_{h-1}\,, \\ D_h & = & -\beta + C_{1,h-1}'(\nu + \Sigma \gamma_o) + \frac{1}{2} C_{1,h-1}'(\Sigma \Sigma') C_{1,h-1} + D_{h-1}\,, \end{array} \right. \tag{7}$$

□ and where :

$$\widetilde{\Phi}^* = \begin{bmatrix} \Phi_1 + \Sigma\gamma_1 & \dots & \dots & \Phi_{p-1} + \Sigma\gamma_{p-1} & \Phi_p + \Sigma\gamma_p \\ I_K & \mathbf{0}_K & \dots & \mathbf{0}_K & \mathbf{0}_K \\ \mathbf{0}_K & I_K & \dots & \mathbf{0}_K & \mathbf{0}_K \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0}_K & \dots & \dots & I_K & \mathbf{0}_K \end{bmatrix} \text{ is a } (Kp, Kp) \text{ matrix}, \tag{8}$$

γ_i's are (K, K) matrices such that $\widetilde{\Gamma} = [\gamma_1, \dots, \gamma_p]$. That is : $\gamma_i = \begin{bmatrix} \widetilde{\gamma}'_{1,i} \\ \vdots \\ \widetilde{\gamma}'_{K,i} \end{bmatrix}$;

□ the initial conditions are $C_0 = 0, D_0 = 0$ (or $C_1 = -\alpha, D_1 = -\beta$), where $C_{1,h}$ indicates the vector of the first K components of the (Kp)-dimensional vector C_h.

□ The (continuously compounded) term structure of interest rates is given by:

$$R(t,t+h) = -\frac{1}{h}\log B(t,t+h) = -\frac{C_h'}{h}X_t - \frac{D_h}{h}\,, \quad h \geq 1\,, \tag{9}$$

□ For a given date t, any yield $R(t,t+h)$ is an affine function of the factor X_t, that is of the p most recent lagged values of the K-dimensional factor x_{t+1}.

□ With regard to the one-period geometric zero-coupon bond return process $\rho = [\,\rho(t,T), 0 \leq\ t \leq T\,]$, it is easy to verify that :

$$\rho(t+1,\,T) \;\;=\;\; r_t - \tfrac{1}{2}\,\omega(t+1,\,T)'\omega(t+1,\,T) \;+\; \omega(t+1,\,T)'\Gamma_t - \omega(t+1,\,T)'\,\varepsilon_{t+1}\,,$$

where $\omega(t+1,\,T) = -(\Sigma' C_{1,T-t-1})$ is an K-dimensional vector.

□ The associated risk premium, between t and $t+1$, is given by :

$$\begin{aligned}\lambda_t(T) &= \omega(t+1,T)'\,\Gamma_t = \textstyle\sum_{i=1}^{K}\omega_i(t+1,T)\,\Gamma_{i,t}\\ &= \textstyle\sum_{i=1}^{K}\omega_i(t+1,T)(\gamma_{o,i}+\sum_{j=1}^{p}\widetilde{\gamma}'_{i,j}x_{t-j+1})\,,\end{aligned}$$

where $\omega(t+1,T) = [\omega_1(t+1,T),\ldots,\omega_K(t+1,T)]'$.

□ One may notice that, in this multivariate setting, the magnitude of $\lambda_t(T)$ is given by a linear combination of the K risk premia $\Gamma_{i,t} = \gamma_{o,i} + \sum_{j=1}^{p}\widetilde{\gamma}'_{i,j}x_{t-j+1}$.

□ Moreover, for a given matrix $\widetilde{\Gamma}$ different from zero, $\lambda_t(T)$ is function of the p most recent lagged values of the K-dimensional factor x_{t+1}.

4.5.2 The Risk-Neutral Dynamics

☐ The risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$, is given by:

$$
\begin{aligned}
E_t^{\mathbb{Q}}[\exp(u'x_{t+1})] &= E_t\left[\frac{M_{t,t+1}}{E_t(M_{t,t+1})}\exp(u'x_{t+1})\right] \\
&= E_t\left[\exp\left(\Gamma_t'\,\varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma_t'\Gamma_t + u'x_{t+1}\right)\right] \\
&= \exp\left[u'(\nu + \Phi X_t) - \tfrac{1}{2}\Gamma_t'\Gamma_t)\right] E_t[\exp(\Gamma_t + \Sigma' u)'\varepsilon_{t+1}] \\
&= \exp\left[u'[(\nu + \Sigma\gamma_o) + (\Phi + \Sigma\widetilde{\Gamma})X_t] + \tfrac{1}{2}u'(\Sigma\Sigma')u\right] .
\end{aligned}
$$

☐ Under the risk neutral probability $\mathbb{Q}$, x_{t+1} is a K-dimensional VAR(p) process of the following type:

$$
\begin{aligned}
x_{t+1} &= \nu^* + \Phi_1^* x_t + \ldots + \Phi_p^* x_{t+1-p} + \Sigma^*\eta_{t+1} \\
&= \nu^* + \Phi^* X_t + \Sigma^*\eta_{t+1}\,,
\end{aligned}
\tag{10}
$$

□ with

$$\nu^* = (\nu + \Sigma\gamma_o), \;\; \Phi_j^* = (\Phi_j + \Sigma\gamma_j), \quad \text{for } j \in \{1, \dots, p\}$$

$$\Phi^* = [\Phi_1^*, \dots, \Phi_p^*], \;\; \Sigma^* = \Sigma,$$

□ where η_{t+1} is (under $\mathbb{Q}$) an K-dimensional gaussian white noise with $\mathcal{N}(0, I_K)$ distribution.

□ This model can be represented in the following vectorial form :

$$X_{t+1} = \widetilde{\Phi}^* X_t + [\nu^* + \Sigma^* \eta_{t+1}] e_1,$$

where e_1 is the vector of size (Kp), with all entries equal to zero except for the first K elements which are all equal to one.

4.5.3 The Gaussian VAR(p) Observable Factor-Based Model

☐ It is like in the previous lecture, with

$$x_t = \begin{bmatrix} R(t, t+h_1) \\ R(t, t+h_2) \\ \vdots \\ R(t, t+h_K) \end{bmatrix}$$

and where $R(t, t+h_1) = R(t, t+1) = r_t$ and $h_1 < h_2 < \ldots < h_K$.

☐ The absence of arbitrage conditions for the K yields in x_t imply :

$$\begin{aligned} (i)\ C_1 &= -e_1\,, \quad D_1 = 0\,, \\ (ii)\ C_{h_j} &= -h_j\, e_{h_j}\,, \quad D_{h_j} = 0\,, \forall\ j \in \{2, \ldots, K\}\,, \end{aligned} \tag{11}$$

where e_{h_j} denotes the h_j^{th} element of the canonical basis in $\mathbb{R}^{Kp}$.

4.5.4 The S-Forward Dynamics

- The S-forward dynamics of the K-dimensional factor x_{t+1} has an VAR(p) representation of the following type:

$$x_{t+1} = \nu_S + \Phi_1^* x_t + \ldots + \Phi_p^* x_{t+1-p} + \Sigma^* \xi_{t+1}\,, \tag{12}$$

- with

$$\nu_S = \nu^* - \Sigma^* \omega(t+1, S)\,,$$

- and where $\xi_{t+1} \sim \mathcal{IIN}(0, I_K)$ under $\mathbb{Q}^{(S)}$.

□ This model can be represented in the following vectorial form :

$$X_{t+1} = \widetilde{\Phi}^* X_t + [\nu_S + \Sigma^* \xi_{t+1}] e_1 \, ,$$

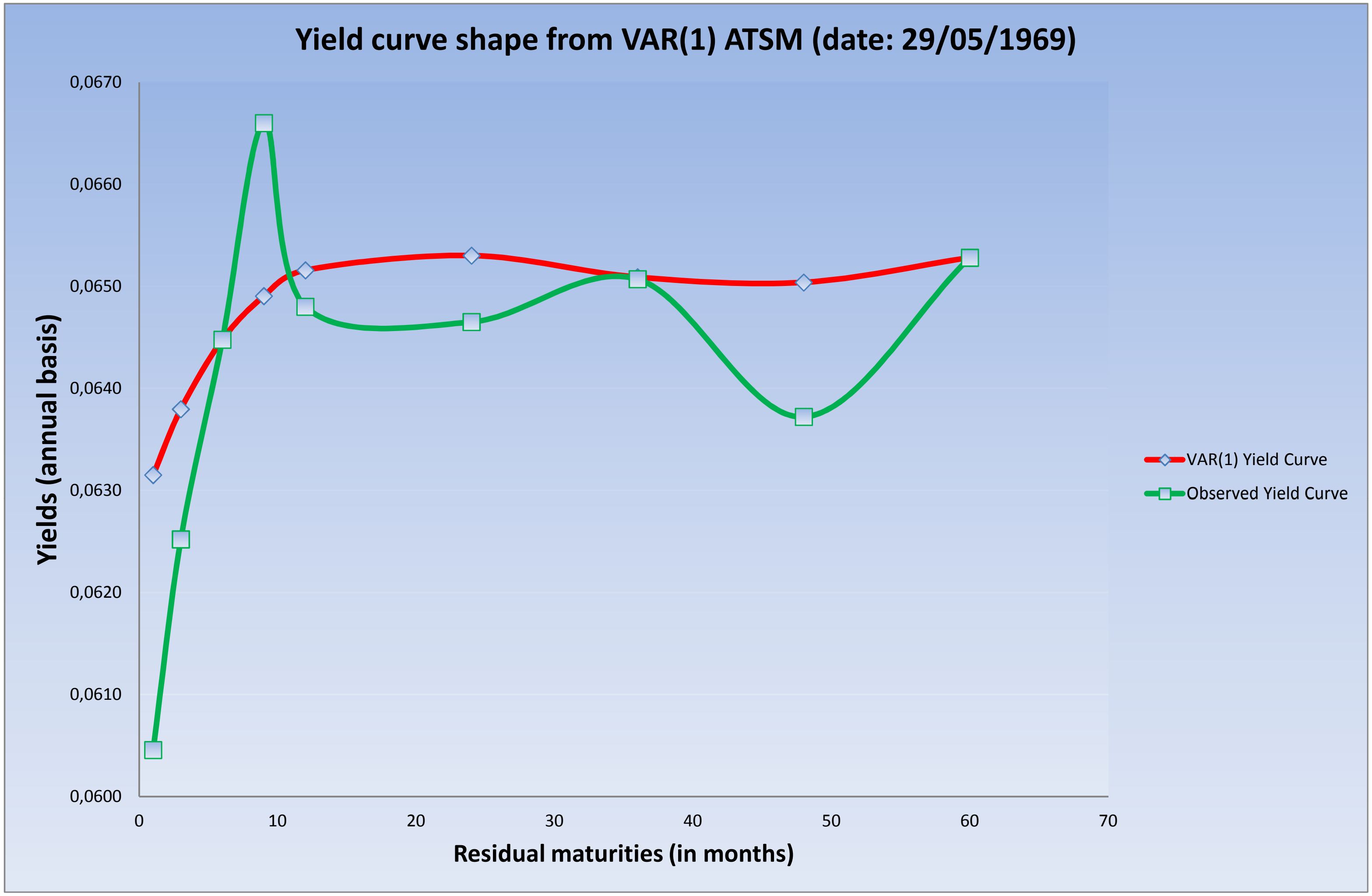
where e_1 denotes the vector of size (Kp), with all entries equal to zero except for the first K elements which are all equal to one.

□ The one-period geometric zero-coupon bond return process is given by:

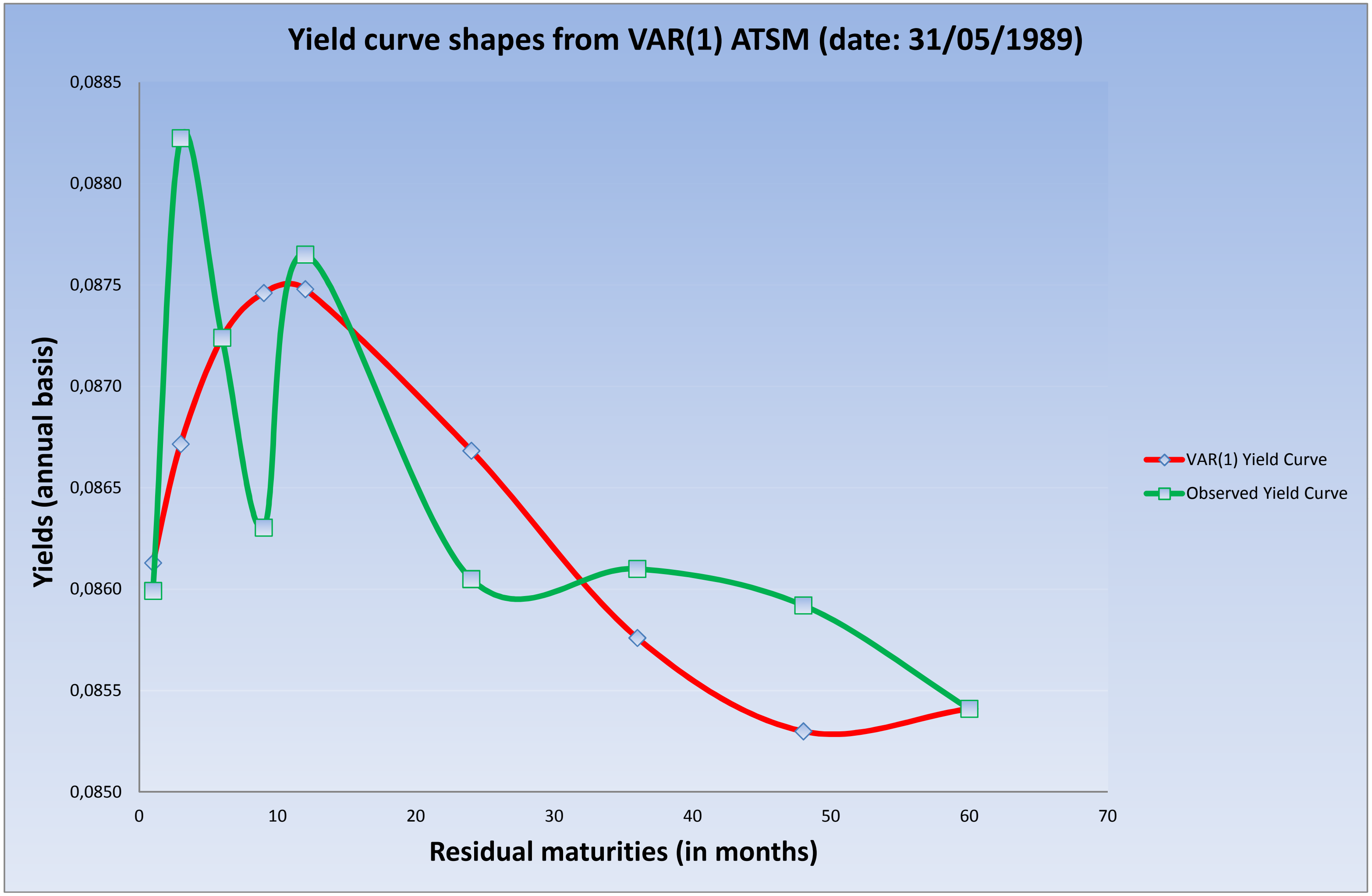
$$\begin{aligned} \rho(t+1, T) &= r_t - \omega(t+1, T)' \, \xi_{t+1} - \tfrac{1}{2} \omega(t+1, T)' \omega(t+1, T) \\ &\quad + \omega(t+1, T)' \omega(t+1, S) \, , \end{aligned}$$

□ with one-period risk premium given by :

$$\lambda_t^{\mathbb{Q}^{(S)}}(T) = \log E_t^{\mathbb{Q}^{(S)}} \exp[\rho(t+1, T)] - r_t = \omega(t+1, T)' \omega(t+1, S) \, .$$



Yield curve shape from VAR(1) ATSM (date: 29/05/1969)
0,0670
0,0660
0,0650
0,0640
0,0630
0,0620
0,0610
0,0600
Yields (annual basis)
0
10
20
30
40
50
60
70
Residual maturities (in months)
VAR(1) Yield Curve
Observed Yield Curve


Yield curve shapes from VAR(1) ATSM (date: 31/05/1989)
0,0885
0,0880
0,0875
0,0870
0,0865
0,0860
0,0855
0,0850
Yields (annual basis)
0
10
20
30
40
50
60
70
Residual maturities (in months)
VAR(1) Yield Curve
Observed Yield Curve