Fixed Income and Credit Risk : exercise sheet n° 04

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Exercise N° 01 [Exponential-affine ZCB Pricing Formula].

Let us consider a discrete-time univariate Gaussian term structure model, in which the factor x_{t+1} has an historical dynamics described by the Gaussian AR(p) process:

$$\begin{aligned} x_{t+1} &= \nu + \varphi_1 x_t + \ldots + \varphi_p x_{t+1-p} + \sigma\varepsilon_{t+1} \\ &= \nu + \varphi' X_t + \sigma\varepsilon_{t+1}, \end{aligned}$$

where ε_{t+1} is a Gaussian white noise with $\mathcal{N}(0,1)$ distribution. We have: $\varphi = [\varphi_1, \ldots, \varphi_p]'$, $X_t = [x_t, \ldots, x_{t+1-p}]'$, and where $\sigma > 0$, ν and φ_i, for $i \in \{1, \ldots, p\}$, are scalar coefficients. Let us also assume that the stochastic discount factor (SDF) $M_{t,t+1}$ for the period $(t, t+1)$ has the following exponential-affine specification:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' X_t + \Gamma_t \varepsilon_{t+1} - \frac{1}{2}\Gamma_t^2\right].$$

Prove that the price at date t of the zero-coupon bond with time to maturity h is :

$$B(t, t+h) = \exp(c_h' X_t + d_h), \quad h \geq 1,$$

where c_h and d_h satisfies the recursive equations :

$$\begin{aligned} c_h &= -\alpha + \Phi' c_{h-1} + c_{1,h-1}\sigma\gamma = -\alpha + \Phi^{*'} c_{h-1}, \\ d_h &= -\beta + c_{1,h-1}(\nu + \gamma_o\sigma) + \tfrac{1}{2}c_{1,h-1}^2\sigma^2 + d_{h-1}, \end{aligned}$$

with :

$$\Phi^* = \begin{bmatrix} \varphi_1 + \sigma\gamma_1 & \ldots & \ldots & \varphi_{p-1} + \sigma\gamma_{p-1} & \varphi_p + \sigma\gamma_p \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & 1 & 0 \end{bmatrix}.$$

and where the initial conditions are $c_0 = 0, d_0 = 0$ (or $c_1 = -\alpha, d_1 = -\beta$); $c_{1,h}$ is the first component of the p-dimensional vector c_h.

Exercise N° 02 [Identification Issue in latent factor Gaussian ATSMs].

Let us consider, for ease of exposition, the Gaussian AR(1) Factor-Based term structure model. This Gaussian ATSM can be summarized as follows:

$$
\begin{aligned}
x_{t+1} &= \nu + \varphi x_t + \sigma\varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim \mathcal{N}(0,1) \;\; (\text{under } \mathbb{P}) \\
M_{t,t+1} &= \exp\left[-\beta - \alpha x_t + \Gamma_t \varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t^2\right]\,, \;\; (\text{SDF}) \\
r_t &= \beta + \alpha x_t\,, \;\; \Gamma_t = \Gamma(x_t) = (\gamma_o + \gamma x_t)\,, \\
R(t,h) &= -\frac{c_h}{h}x_t - \frac{d_h}{h}\,, \\
c_h &= -\alpha + \varphi c_{h-1} + c_{h-1}\sigma\gamma = -\alpha + (\varphi + \sigma\gamma)c_{h-1}\,, \\
d_h &= -\beta + c_{h-1}(\nu + \gamma_o\sigma) + \tfrac{1}{2}c_{h-1}^2\sigma^2 + d_{h-1}\,, \\
& c_0 = 0, d_0 = 0\,.
\end{aligned}
$$

The purpose of this exercise is to discuss the identification problem characterizing the above summarized Gaussian AR(1) ATSM where x_t is a latent factor. Show that there exist different set of historical parameter values (ν, φ, σ) and/or parameter values in the SDF $(\beta, \alpha, \gamma_o, \gamma)$ generating the same yield for any residual maturity. Propose a set of parameter restrictions giving the possibility to solve this identification problem.

Exercise N° 03 [Excess Returns of Zero-Coupon Bonds].

Prove that, under the absence of arbitrage opportunity, and for a fixed maturity T, the one-period geometric zero-coupon bond return process $\rho = [\,\rho(t,T), 0 \leq t \leq T\,]$, where $\rho(t+1,T) = \log[B(t+1,\,T)] - \log[B(t,\,T)]$, is given by:

$$\rho(t+1,\,T) \;=\; r_t - \tfrac{1}{2}\,\omega(t+1,\,T)^2 + \omega(t+1,\,T)\Gamma_t - \omega(t+1,\,T)\,\varepsilon_{t+1}\,,$$

where $\omega(t+1,\,T) = -(\sigma c_{1,T-t-1})$.

Exercise N° 04 [Risk-Neutral Laplace Transform of the Gaussian AR(p) Factor].

Let us consider the Gaussian AR(p) distributed factor x_{t+1} and the SDF $M_{t,t+1}$ mentioned in the Exercise 1. Calculate the risk-neutral Laplace transform of x_{t+1}, conditionally to $\underline{x_t}$.

Exercise N° 05 [Risk-Neutral Zero-Coupon Bond Return Process].

Prove that, in the risk-neutral framework, for a fixed maturity T, the one-period geometric zero-coupon bond return process satisfies the relation:

$$\rho(t+1,\,T) = r_t - \tfrac{1}{2}\,\omega(t+1,\,T)^2 - \omega(t+1,\,T)\,\eta_{t+1}\,,$$

with a risk premium equal to :

$$\lambda_t^{\mathbb{Q}}(\rho,1) = \log E_t^{\mathbb{Q}} \exp\left[\rho(t+1,\,T)\right] - r_t = 0\,.$$

Exercise N° 06 [Yield Curve Shapes, Risk-Neutral Stationarity and Long Rates].

Let us consider our discrete-time univariate Gaussian term structure model, in which the factor x_{t+1} has an historical dynamics described by a Gaussian AR(1) process, and the one-period SDF has the exponential-affine form mentioned in the Exercise 1. Assuming $x_{t+1} = r_{t+1}$ (the scalar factor is the short rate), study the kind of shapes the yield curve formula $R(t, h)$ can generate on the basis of the dynamic properties of the system of difference equations (c_h, d_h). Then, repeat the same analysis for $p = 2$.

Exercise N° 07 [Exercise N° 06, continued].

Let us now consider our discrete-time univariate Gaussian term structure model, in which the factor x_{t+1} has an historical dynamics described by a Gaussian AR(p) process, and the one-period SDF has the exponential-affine form mentioned in the Exercise 1. Study in this general case the dynamic properties of the system of difference equations (c_h, d_h) and provide a generalization of the results shown in the Exercise N° 05. In addition, derive the equation of the "long rate" $R(t, +\infty)$.

Exercise N° 08 [Gaussian AR(p) Factor Dynamics under the S-Forward probability].

Let us now consider our discrete-time univariate Gaussian term structure model, in which the factor x_{t+1} has an historical dynamics described by a Gaussian AR(p) process, and the one-period SDF has the exponential-affine form mentioned in the Exercise 1. Calculate the S-Forward dynamics of x_{t+1}, conditionally to $\underline{x_t}$.

Exercise N° 09 [Zero-Coupon Bond Return Process under the S-Forward probability].

Prove that, under the S-Forward probability $\mathbb{Q}^{(S)}$, the one-period geometric zero-coupon bond return process is described by the relation:

$$\rho(t+1,\, T) \quad = \quad r_t - \tfrac{1}{2}\,\omega(t+1,\, T)^2 + \omega(t+1,\, T)\omega(t+1,\, S) - \omega(t+1,\, T)\,\xi_{t+1}\,,$$

with a one-period risk premium given by :

$$\lambda_t^{\mathbb{Q}^{(S)}}(T) = \log E_t^{\mathbb{Q}^{(S)}} \exp\left[\rho(t+1,\, T)\right] - r_t = \omega(t+1,\, T)\omega(t+1,\, S)\,.$$

Exercise N$^\circ$ 10 [No-arbitrage restrictions for the short rate and spread].

Let us assume to have a bivariate Gaussian VAR(1) Factor-Based term structure models, and let us assume that the factor x_t be given by $x_t = (r_t, S_t)'$ where $r_t = R(t, t+1)$ is the yield with the shortest maturity in our data base (it is the short rate) and where $S_t = R_t - r_t$ is the spread between the long rate R_t (the yield with the longest maturity in our data base) and the short rate. This Gaussian VAR(1) ATSM can be summarized as follows:

$$
\begin{aligned}
x_{t+1} &= \nu + \Phi x_t + \Sigma \varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \sim \mathcal{N}(0, I_2) \;\; \text{(under } \mathbb{P}) \\
M_{t,t+1} &= \exp\left[-\beta - \alpha' x_t + \Gamma_t' \varepsilon_{t+1} - \tfrac{1}{2}\Gamma_t'\Gamma_t\right]\,, \;\; \text{(SDF)} \\
\Gamma_t &= \Gamma(x_t) = (\gamma_o + \gamma\, x_t)\,, \\
R(t, t+h) &= -\frac{{C_h}'}{h} x_t - \frac{D_h}{h}\,, \\
C_h &= -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} = -\alpha + {\Phi^*}' C_{h-1}\,, \\
D_h &= -\beta + C_{h-1}'(\nu + \Sigma\gamma_o) + \tfrac{1}{2} C_{h-1}'(\Sigma\Sigma') C_{h-1} + D_{h-1}\,, \\
& C_0 = 0, D_0 = 0\,.
\end{aligned}
$$

Write the complete set of no-arbitrage restrictions that this model has to satisfy.