Fixed Income and Credit Risk : solutions for exercise sheet n° 03

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Exercise N° 01 [Replicating strategies in the one-period model].

We have a one-period financial market $\{S(0), \mathbf{S}\}$ where $S(0) \in \mathbb{R}_+^{d+1}$, $S(0) = [S_0(0), S_1(0), \ldots, S_d(0)]'$, is the date $t = 0$ vector of basic asset prices. $\mathbf{S}$ denotes the $[(d+1) \times N]$-matrix of payoffs.

a) $k = rank(\mathbf{S}') = N = d + 1$ (complete markets without redundant assets).

$\mathbf{S}$ is a square full-rank matrix and therefore the inverse $(\mathbf{S}')^{-1}$ exists. The system $y = \mathbf{S}'\varphi$ admits always a unique replicating strategy (i.e. hedging portfolio) $\varphi = (\mathbf{S}')^{-1}y$ for any $y \in \mathbb{R}^{d+1}$.

b) $k = rank(\mathbf{S}') = N < d + 1$ (complete markets with redundant assets).

Let us divide the payoff matrix $\mathbf{S}'$ into a (N, N)-matrix $\mathbf{S_1}'$ of linearly independent payoffs and a $(N, d+1-N)$-matrix $\mathbf{S_2}'$ of redundant payoffs. We have $\mathbf{S}' = (\mathbf{S_1}' : \mathbf{S_2}')$.

Let us also divide the portfolio $\varphi \in \mathbb{R}^{d+1}$ in a N-dimensional vector φ_1 associated to the linearly independent assets and a $(d+1-N)$-dimensional vector φ_2 associated to the linearly dependent assets. We have $\varphi = (\varphi_1', \varphi_2')'$.

Given that $\mathbf{S_1}'$ is a square matrix of full rank ($k = rank(\mathbf{S_1}') = N$), then $(\mathbf{S_1}')^{-1}$ exists. Moreover, given that $\mathbf{S_2}'$ contains the (redundant) payoffs that can be replicated by the (non-redundant) payoffs in $\mathbf{S_1}'$, there exists a $(N, d+1-N)$ matrix C such that $\mathbf{S_2}' = \mathbf{S_1}'C$.

For any payoff $y \in \mathbb{R}^N$ we can write $y = \mathbf{S}'\varphi = \mathbf{S_1}'\varphi_1 + \mathbf{S_2}'\varphi_2 = \mathbf{S_1}'[\varphi_1 + C\varphi_2]$ and, given that $(\mathbf{S_1}')^{-1}$ exists we have that the formula giving the replicating strategy is $\varphi_1 = (\mathbf{S_1}')^{-1}y - C\varphi_2$. We observe that the implementation of that strategy requires first to (arbitrarily) choose (to fix) the portfolio φ_2 of redundant assets. This means that we have $(d+1-N)$ free parameters indicating the multiplicity of the replicating strategies φ_1 for the same payoff y.

c) $k = rank(\mathbf{S}') = d + 1 < N$ (incomplete markets without redundant assets).

Given that $k = rank(\mathbf{S}') = d+1$, then $(\mathbf{S}\mathbf{S}')^{-1}$ exists and the right inverse of $\mathbf{S}$ is well defined. For a given payoff $y \in \mathbb{R}^N$ let us, first, multiplying (on the left) system $y = \mathbf{S}'\varphi$ by $\mathbf{S}$ and then by $(\mathbf{S}\mathbf{S}')^{-1}$. We obtain $\varphi = L^{(S')}y$, where $L^{(S')} = (\mathbf{S}\mathbf{S}')^{-1}\mathbf{S}$ is the transposed of the right inverse of $\mathbf{S}$.

The portfolio $\varphi = L^{(S')}y$ is a solution of the modified system $\mathbf{S}y = \mathbf{S}\mathbf{S}'\varphi$ and not (in general) of the original system $y = \mathbf{S}'\varphi$ (we are interested in).

Now, if $rank(\mathbf{S}') = rank(\mathbf{S}' : y)$, the portfolio $\varphi = L^{(S')}y$ is the unique replicating strategy for the payoff y. This means that, we have a (unique) solution when the payoff $y \in \mathbb{R}^N$ is redundant or, in other words, when $y \in \mathcal{M}(\mathbf{S})$ (when the payoff belongs to the asset span).

If the payoff $y \in \mathbb{R}^{d+1}$ and the basic assets are linearly independent ($y \notin \mathcal{M}(\mathbf{S})$), then $\varphi = L^{(S')}y$ is not a solution of the original system $y = \mathbf{S}'\varphi$ and we are not able to build a replicating strategy for y.

d) $k = rank(\mathbf{S}') < d+1$ and $k < N$ (incomplete markets with redundant assets).

Let us divide the payoff matrix $\mathbf{S}'$ into a (N,k)-matrix $\mathbf{\bar{S}_1'}$ of linearly independent payoffs and a $(N, d+1-k)$-matrix $\mathbf{\bar{S}_2'}$ of redundant payoffs. We have $\mathbf{S}' = (\mathbf{\bar{S}_1'} : \mathbf{\bar{S}_2'})$.

Let us also divide the portfolio $\varphi \in \mathbb{R}^{d+1}$ in a k-dimensional vector $\bar{\varphi}_1$ associated to the linearly independent assets and a $(d+1-k)$-dimensional vector $\bar{\varphi}_2$ associated to the linearly dependent assets. We have $\varphi = (\bar{\varphi}_1', \bar{\varphi}_2')'$.

The matrix $\mathbf{\bar{S}_1'}$ is not square but $(\mathbf{\bar{S}_1}\mathbf{\bar{S}_1'})$ is invertible. Moreover, given that $\mathbf{\bar{S}_2'}$ contains the (redundant) payoffs that can be replicated by the (non-redundant) payoffs in $\mathbf{\bar{S}_1'}$, there exists a $(N, d+1-k)$ matrix $\bar{C}$ such that $\mathbf{\bar{S}_2'} = \mathbf{\bar{S}_1'}\bar{C}$.

For a given payoff $y \in \mathbb{R}^N$ we can, first, write the original system as $y = \mathbf{S}'\varphi = \mathbf{\bar{S}_1'}\bar{\varphi}_1 + \mathbf{\bar{S}_2'}\bar{\varphi}_2 = \mathbf{\bar{S}_1'}[\bar{\varphi}_1 + \bar{C}\bar{\varphi}_2]$ and, then, we can considers the associated modified system $\mathbf{\bar{S}_1}y = \mathbf{\bar{S}_1}\mathbf{\bar{S}_1'}[\bar{\varphi}_1 + \bar{C}\bar{\varphi}_2]$.

Now, given that $(\mathbf{\bar{S}_1}\mathbf{\bar{S}_1'})^{-1}$ exists, we have that the formula giving the solution to the modified system is $\bar{\varphi}_1 = (\mathbf{\bar{S}_1}\mathbf{\bar{S}_1'})^{-1}\mathbf{\bar{S}_1}\, y - \bar{C}\bar{\varphi}_2 = L^{(\bar{S}_1')}y - \bar{C}\bar{\varphi}_2$. We observe that the implementation of that strategy requires first to (arbitrarily) choose (to fix) the portfolio $\bar{\varphi}_2$ of redundant assets. This means that we have $(d+1-k)$ free parameters indicating the multiplicity of solutions $\bar{\varphi}_1$ (for the modified system).

Now, if $rank(\mathbf{S}') = rank(\mathbf{S}' : y)$, the portfolio $\bar{\varphi}_1 = L^{(\bar{S}_1')}y - \bar{C}\bar{\varphi}_2$ identifies the infinitely many replicating strategies for the payoff y. This means that, we have a (non unique) solution when the payoff $y \in \mathbb{R}^N$ is redundant or, in other words, when $y \in \mathcal{M}(\mathbf{S})$ (when the payoff belongs to the asset span generated by the payoff in $\mathbf{\bar{S}_1}$).

If the payoff $y \in \mathbb{R}^N$ and the payoffs in $\mathbf{\bar{S}_1}$ are linearly independent ($y \notin \mathcal{M}(\mathbf{S})$), then $\bar{\varphi}_1$ is not a solution of the original system $y = \mathbf{S}'\varphi$ and the replicating strategy for y does not exists.

Exercise N° 02 [Complete financial markets].

If the financial market $\{S(0), \mathbf{S}\}$ is complete, then $k = rank(\mathbf{S}') = N$ and we can consider two possible cases depending on the presence or not of redundant assets, given that we always have $k \leq (d+1)$. This means that we are in a situation where $k = N \leq (d+1)$. This condition guarantee the existence of the left inverse of $\mathbf{S}$, namely $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\,\mathbf{S}'$.

If the payoff matrix $\mathbf{S}$ admits left inverse $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\,\mathbf{S}'$, then we have $(d+1) \geq N$ and the N columns of $\mathbf{S}$ are linearly independent. Thus, we have $N = rank(\mathbf{S})$ and therefore the market is complete.

Exercise N$^\circ$ 03 [The Law of One Price].

The law of one price (LOP) states that if $\mathbf{S}'\varphi^* = \mathbf{S}'\varphi^{**}$, with $\varphi^* \neq \varphi^{**}$ and $\varphi^*, \varphi^{**} \in \mathbb{R}^{d+1}$, then $S_{\varphi^*}(0) = S_{\varphi^{**}}(0)$.

a) If the LOP holds then $q(.)$ is single-valued. We have to prove that if the LOP holds then $q(.)$ is a linear function (of the asset span). To prove the linearity let us consider two payoffs y^* and y^{**} both in the asset span. We have therefore $y^* = \mathbf{S}'\varphi^*$, $\varphi^* \in \mathbb{R}^{d+1}$ with given price $S_{\varphi^*}(0)$, and $y^{**} = \mathbf{S}'\varphi^{**}$, $\varphi^{**} \in \mathbb{R}^{d+1}$ with given price $S_{\varphi^{**}}(0)$. By definition of payoff pricing function $q(y^*) = S_{\varphi^*}(0)$ (the price of the portfolio generating y^*) and $q(y^{**}) = S_{\varphi^{**}}(0)$ (the price of the portfolio generating y^*).

Now, let us take arbitrary real numbers $\lambda_1, \lambda_2 \in \mathbb{R}$. It is clear that the payoff $(\lambda_1 y^* + \lambda_2 y^{**})$ can be generated by the strategy $(\lambda_1\varphi^* + \lambda_2\varphi^{**})$: $\mathbf{S}'[\lambda_1\varphi^* + \lambda_2\varphi^{**}] = \lambda_1\mathbf{S}'\varphi^* + \lambda_2\mathbf{S}'\varphi^{**} = \lambda_1 y^* + \lambda_2 y^{**}$. The value (price) of that portfolio is $\lambda_1 S_{\varphi^*}(0) + \lambda_2 S_{\varphi^{**}}(0)$. Given that $q(.)$ is single-valued, for any given $\lambda_1, \lambda_2 \in \mathbb{R}$, the price of the payoff $(\lambda_1 y^* + \lambda_2 y^{**})$ is always given by $q(\lambda_1 y^* + \lambda_2 y^{**}) = \lambda_1 S_{\varphi^*}(0) + \lambda_2 S_{\varphi^{**}}(0)$, that is the value of the replicating portfolio. The right-hand side of the last relation equals $(\lambda_1 q(y^*) + \lambda_2 q(y^{**}))$, and thus $q(.)$ is linear.

b) We have to prove that, if the payoff pricing function $q(.)$ is linear then the LOP holds. Let us consider the two payoffs y^* and y^{**} both in the asset span. By linearity we have: $q(\lambda_1 y^* + \lambda_2 y^{**}) = \lambda_1 q(y^*) + \lambda_2 q(y^{**}) = \lambda_1 S_{\varphi^*}(0) + \lambda_2 S_{\varphi^{**}}(0)$, for any $\lambda_1, \lambda_2 \in \mathbb{R}$. In particular, for $\lambda_1 = \lambda_2 = 0$, we have $q(0) = 0$.

Now, let us assume that $y^* = \mathbf{S}'\varphi^*$ and $y^{**} = \mathbf{S}'\varphi^{**}$ are such that $\mathbf{S}'\varphi^* = \mathbf{S}'\varphi^{**}$. This means that $y^* - y^{**} = \mathbf{S}'(\varphi^* - \varphi^{**}) = 0$ (a zero payoff). By linearity, we can always write $q(y^* - y^{**}) = q(y^*) - q(y^{**}) = S_{\varphi^*}(0) - S_{\varphi^{**}}(0)$. At the same time we have $q(y^* - y^{**}) = q(0) = 0$. Thus $S_{\varphi^*}(0) - S_{\varphi^{**}}(0) = 0$, i.e. $S_{\varphi^*}(0) = S_{\varphi^{**}}(0)$, and the LOP holds.

Exercise N$^\circ$ 04 [Pricing payoffs in the asset span].

a) Let us consider an incomplete market without redundant assets ($k = rank(\mathbf{S}') = d + 1 < N$), a payoff $y \in \mathcal{M}(\mathbf{S})$ and the associated system $\mathbf{S}'\varphi = y$. Given that $rank(\mathbf{S}') = d + 1$ we have that $(\mathbf{S}\mathbf{S}')^{-1}$ exists and the right inverse of $\mathbf{S}$ is well defined : $R^{(S)} = \mathbf{S}'(\mathbf{S}\mathbf{S}')^{-1}$. This means that we also have $(R^{(S)})' = (\mathbf{S}\mathbf{S}')^{-1}\mathbf{S} = L^{(S')}$. Thus, the replicating strategy is $\varphi = L^{(S')}y = (\mathbf{S}\mathbf{S}')^{-1}\mathbf{S}y$ which a solution of the original system being $y \in \mathcal{M}(\mathbf{S})$.

Now, the price of the payoff $y \in \mathcal{M}(\mathbf{S})$ is the value of the replicating portfolio an therefore $q(y) = S(0)'\varphi = S(0)'L^{(S')}y = S(0)'(\mathbf{S}\mathbf{S}')^{-1}\mathbf{S}y = y'R^{(S)}S(0)$ and the pricing formula is proved.

b) We have an incomplete market with redundant assets ($k < d+1$, $k < N$). Let us denote with $\bar{\mathbf{S}}$ the (k, N) payoff matrix of the no redundant assets and with $\bar{S}(0)$ the vector of these k asset prices. Given that $rank(\mathbf{S}') = k$ we have that $(\bar{\mathbf{S}}\bar{\mathbf{S}}')^{-1}$ exists and the right inverse of $\bar{\mathbf{S}}$ is well defined : $R^{(\bar{S})} = \bar{\mathbf{S}}'(\bar{\mathbf{S}}\bar{\mathbf{S}}')^{-1}$. This means that we also have $(R^{(\bar{S})})' = (\bar{\mathbf{S}}\bar{\mathbf{S}}')^{-1}\bar{\mathbf{S}} = L^{(\bar{S}')}$. Following the same steps as above, we find $q(y) = y'\, R^{(\bar{S})}\, \bar{S}(0) = \bar{S}(0)'\, L^{(\bar{S}\,')}\, y$ and the pricing formula is proved.

Exercise N° 05 [First Fundamental Theorem of Asset Pricing].

We have to prove that in the financial market $\{S(0), \mathbf{S}\}$ there are no arbitrage opportunities if and only if there exists a strictly positive vector of state prices $q^{(ad)} \in \mathbb{R}^N_{++}$ such that:

$$S(0) = \mathbf{S}\, q^{(ad)} .$$

a) If there exists a (not unique in general) vector $q^{(ad)} \in \mathbb{R}^N_{++}$ such that $S(0) = \mathbf{S}\, q^{(ad)}$, then for any portfolio $\varphi \in \mathbb{R}^{d+1}$ we have $S_\varphi(0) = \varphi' S(0) = \varphi' \mathbf{S}\, q^{(ad)}$. Now, if $\varphi' \mathbf{S} \geq 0$ then $\varphi' S(0) \geq 0$ given that $q^{(ad)} \in \mathbb{R}^N_{++}$. If $\varphi' \mathbf{S} > 0$ then $\varphi' S(0) > 0$ given that $q^{(ad)} \in \mathbb{R}^N_{++}$. Thus, the absence of arbitrage opportunity (AAO) principle is satisfied.

b) We have to prove that, given the financial market $\{S(0), \mathbf{S}\}$, under the AAO principle, then there exists a (not unique in general) vector $q^{(ad)} \in \mathbb{R}^N_{++}$ such that $S(0) = \mathbf{S}\, q^{(ad)}$.

Let $\mathcal{M}(S(0), \mathbf{S})$ the market span defined as:

$$\mathcal{M}(S(0), \mathbf{S}) = \Big\{ (x, y')' \,, x \in \mathbb{R} \,, y \in \mathbb{R}^N \,:\, x = -S(0)' \varphi \,, y = \mathbf{S}' \varphi \,, \varphi \in \mathbb{R}^{d+1} \Big\} \subseteq \mathbb{R}^{N+1} .$$

Let us introduce the positive orthant of $\mathbb{R}^{N+1}$

$$\mathbb{R}^{N+1}_+ = \big\{ z \in \mathbb{R}^{N+1} \,:\, z_j \,\geq\, 0 \;\forall 1 \leq j \leq N+1 \,; \exists\, j \,:\, z_j \,>\, 0 \big\} \;,$$

and the unit simplex of $\mathbb{R}^{N+1}$: $\Delta^N := \Big\{ z \in \mathbb{R}^{N+1}_+ \,:\, \sum_{j=1}^{N+1} z_j = 1 \Big\}$.
The AAO principle implies that all the elements of the vector $(x, y')' \in \mathcal{M}(S(0), \mathbf{S})$ cannot be positive. Thus, if we assume that the market satisfies the AAO principle, then we have $\mathcal{M}(S(0), \mathbf{S}) \bigcap \mathbb{R}^{N+1}_+ = \{0\}$. This is also true for any compact subset of $\mathbb{R}^{N+1}_+$, namely the unit simplex of $\mathbb{R}^{N+1}$. Thus, we also have $\mathcal{M}(S(0), \mathbf{S}) \bigcap \Delta^N = \phi$. This result naturally suggest (for our purpose) the use of the following:

Theorem (The Minkowski Separation Theorem): Let A and B be two non-empty convex subsets of $\mathbb{R}^s$, where A is closed, B is compact and $A \bigcap B = \phi$. Then, there exists a vector of non-zero coefficients $\psi = (\psi_1, \ldots, \psi_s)'$ and two distinct numbers b_1 and b_2 such that:

$$\forall a \in A \,,\;\; \forall b \in B \,,\;\; a' \psi \leq b_1 < b_2 \leq b' \psi \,.$$

In other words, there exists a non-zero linear functional $F : \mathbb{R}^s \mapsto \mathbb{R}$, $F(c) = c' \psi$ with $\psi \neq 0$, such that:

$$\forall a \in A \,,\;\; \forall b \in B \,,\;\; F(a) \leq b_1 < b_2 \leq F(b) \,.$$

In our problem $\mathcal{M}(S(0), \mathbf{S})$ is a closed and convex subset of $\mathbb{R}^{N+1}$ and Δ^N is a compact and convex subset of $\mathbb{R}^{N+1}$. The Minkowski Separation Theorem guarantee the existence of a vector of non-zero coefficients $\psi = (\psi_0, \ldots, \psi_N)' \in \mathbb{R}^{N+1}$ and two distinct numbers such that:

$$\forall \alpha \in \mathcal{M}(S(0), \mathbf{S}) \,,\;\; \forall \sigma \in \Delta^N \,,\;\; F(\alpha) = \alpha' \psi \leq b_1 < b_2 \leq \sigma' \psi = F(\sigma) \,.$$

Given that $0 \in \mathcal{M}(S(0), \mathbf{S})$ and $0 \notin \Delta^N$, we have: $F(0) = 0 \leq b_1 < b_2 \leq \sigma'\psi = F(\sigma)$ for all $\sigma \in \Delta^N$, thus we find $b_1 \geq 0$. Now if we move along the unit simplex boundaries and we successively choose the vectors $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)' \in \Delta^N$ of the canonical basis in $\mathbb{R}^{N+1}$, then we find $0 \leq b_1 < b_2 \leq \psi_j$ for all $j \in \{0, \ldots, N\}$. Thus, we have found a vector $\psi \in \mathbb{R}^{N+1}_{++}$. Without loss of generality, let us assume $\psi_0 = 1$ and let us denote $\psi = (1, \bar{\psi}')' = (1, \psi_1, \ldots, \psi_N) \in \mathbb{R}^{N+1}_{++}$.

Now, taking into account the form of the elements $\alpha \in \mathcal{M}(S(0), \mathbf{S})$, we can write $\alpha_0 + \sum_{j=1}^N \alpha_j \psi_j \leq 0$. This inequality can be written as $(-S(0) + \mathbf{S}\bar{\psi})'\varphi \leq 0$. Hence, $S(0) = \mathbf{S}\bar{\psi}$ (any vector $\psi = (1, \bar{\psi}')' \in \mathbb{R}^{N+1}_{++}$ is orthogonal to $\mathcal{M}(S(0), \mathbf{S})$) and $\bar{\psi} = q^{(ad)} \in \mathbb{R}^N_{++}$ is our vector of positive state prices.

Exercise N° 06.

(i) The price of the risk-free bond is $S_0(t = 0) = \sum_{j=1}^{5} q_j^{ad} = 0.9803$ and its continuously compounded interest rate is $r = \ln(1/0.9803) = 0.0199$.

(ii) The any risk-neutral q_j, for any $j \in \{1, \ldots, 5\}$, is given by $q_j = \dfrac{q_j^{ad}}{\sum_{j=1}^5 q_j^{ad}}$. This means that $q_1 = \frac{0.1255}{0.9803} = 0.1280$, $q_2 = 0.2500$, $q_3 = 0.373$, $q_4 = 0.062$ and $q_5 = 0.187$.

(iii) The price of the new asset is given by $S_\alpha(t = 0) = q^{ad'} S_\alpha(t = 1) = 3.7375$.

Exercise N° 07.

$i)$ Given that $k = rank(\mathbf{S}') = 3 = N$, where N denotes the number of states of nature, then the market is complete. We can equivalently say that the market is complete because $\mathbf{S}$ admits left inverse $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\, \mathbf{S}'$.

$ii)$ Given that $k = rank(\mathbf{S}') = 3 = N < d + 1 = 4$, where $d + 1 = 4$ denotes the number of assets, in this market we have $d + 1 - N = 1$ redundant asset, that is an asset with a payoff that can be replicated by a linear combination of the three other assets. This result can be formally motivated by the fact that the payoff matrix $\mathbf{S}$ has only left inverse $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\, \mathbf{S}'$.

$iii)$ The first fundamental theorem of asset pricing tell us that the financial market $\{S(0), \mathbf{S}\}$ is arbitrage-free if and only if there exists a vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ of state prices such that $S(0) = \mathbf{S}q^{ad}$. Given that the market is complete, then the unique solution is $q^* = q^{ad} = L^{(S)}\, S(0) = (0.2, 0.6, 0.2)'$. We have $q^* \in \mathbb{R}^3_{++}$ and thus the market is arbitrage-free.

Exercise N° 08.

i) the (continuously compounded) short rate return of the risk-free asset is given by $r = \ln(3/S_0(0)) = \ln(3/2) = 0.4055$.

ii) The solution to $S(0) = \mathbf{S}\, q^{ad}$ is:

$$\begin{aligned} q_1^{ad} &= \frac{1}{3} - q_4^{ad}\,, \\ q_2^{ad} &= \frac{1}{6}\,, \\ q_3^{ad} &= \frac{1}{6}\,, \end{aligned}$$

and q_4^{ad} arbitrary. From the first fundamental theorem of asset pricing we know that there is no arbitrage if and only if $q^{ad} \in \mathbb{R}^4_{++}$. This condition is clearly satisfied for any $q_4^{ad} \in]0, \frac{1}{3}[$. Thus, for any $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad}, q_4^{ad})' = \left(\frac{1}{3} - q_4^{ad}, \frac{1}{6}, \frac{1}{6}, q_4^{ad}\right)'$, with $q_4^{ad} \in]0, \frac{1}{3}[$, the market is arbitrage free by the first fundamental theorem of asset pricing.

iii) From the second fundamental theorem of asset pricing we know that, an arbitrage free market is complete if and only if the solution $q^{ad} \in \mathbb{R}^4_{++}$ to $S(0) = \mathbf{S}\, q^{ad}$ is unique.

We have seen from *ii*) that this is not the case given that any $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad}, q_4^{ad})' = \left(\frac{1}{3} - q_4^{ad}, \frac{1}{6}, \frac{1}{6}, q_4^{ad}\right)'$, with $q_4^{ad} \in]0, \frac{1}{3}[$, is in $\mathbb{R}^4_{++}$. Thus, the market is not complete. We can also say that the market is not complete given that we have $N = 4$ states of the nature and only $d + 1 = 3$ assets in the market.

iv) We know that $e^r = 1/\sum_{j=1}^4 q_j^{ad}$, and thus $q_0^{ad} := \sum_{j=1}^4 q_j^{ad} = 2/3$. We also know that any element q_j in the 4-dimensional vector $q = (q_1, q_2, q_3, q_4)'$ of risk-neutral probabilities is given by $q_j = q_j^{ad}/q_0^{ad}$. In our case, this implies that the risk-neutral probabilities are:

$$q_1 = \frac{1}{2} - q_4\,, \;\; q_2 = \frac{1}{4}\,, \;\; q_3 = \frac{1}{4}\,,$$

and $q_4 \in]0, \frac{1}{2}[$ arbitrary.

v) The no-arbitrage prices of this new asset are:

$$\begin{aligned} S_3(0) &= 1 \times q_1^{ad} + 2 \times q_2^{ad} + 0 \times q_3^{ad} + 1 \times q_4^{ad} \\ &= \frac{1}{3} - q_4^{ad} + \frac{1}{3} + q_4^{ad} = \frac{2}{3}\,. \end{aligned}$$

We are in a incomplete market, and this new asset has a unique no-arbitrage price, that is, it is the same for any $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad}, q_4^{ad})' = \left(\frac{1}{3} - q_4^{ad}, \frac{1}{6}, \frac{1}{6}, q_4^{ad}\right)'$, with $q_4^{ad} \in]0, \frac{1}{3}[$. This means that this new asset is redundant.

Exercise N° 09 [Numeraire invariance of the self-financing trading strategy].

Let us first prove the result for any numeraire N_t. N_t is, by definition, a non-dividend-paying price process and therefore is strictly positive for all $t \in \{0, \ldots, T\}$. In particular, we have $N_0 = 1$. Given the positivity of N_t, we have the following equivalence, which implies the claim:

$$\begin{aligned}
\varphi(t)' S(t) &= \varphi(t+1)' S(t)\,, \;\; t \in \{1, \ldots, T-1\}\,, \\
\Leftrightarrow \varphi(t)' \frac{S(t)}{N(t)} &= \varphi(t+1)' \frac{S(t)}{N(t)}\,, \;\; t \in \{1, \ldots, T-1\}\,.
\end{aligned}$$

The sentence of the exercise is proved assuming $N_t = S_0(t) = \exp(r_0 + \ldots + r_{t-1})$.

Exercise N° 10 [Discounted value process and self-financing trading strategy].

a) Let us assume that $\varphi \in \Phi$. Then, using the definition of self-financing trading strategy ($\varphi(t)'S(t) = \varphi(t+1)'S(t)$), the numeraire invariance theorem (see exercise 6) and the factor that $S_0(0) = 1$ we have:

$$\begin{aligned}
\tilde{S}_\varphi(t) &= \tilde{S}_\varphi(0) + \tilde{G}_\varphi(t) \\
&= S_\varphi(0) + \tilde{G}_\varphi(t) \\
&= \varphi(1)' S(0) + \sum_{\tau=1}^{t} \varphi(\tau)' (\tilde{S}(\tau) - \tilde{S}(\tau - 1)) \\
&= \varphi(1)' \tilde{S}(0) + \varphi(t)' \tilde{S}(t) + \sum_{\tau=1}^{t-1} (\varphi(\tau) - \varphi(\tau+1))' \tilde{S}(\tau) - \varphi(1)' \tilde{S}(0) \\
&= \varphi(t)' \tilde{S}(t)\,,
\end{aligned}$$

and the result is proved.

b) Let us assume now that $\tilde{S}_\varphi(t) = S_\varphi(0) + \tilde{G}_\varphi(t)$ holds true for all $t \in \{0, \ldots, T\}$. By the numeraire invariance theorem it is enough to show the discounted version of $\varphi(t)'S(t) = \varphi(t+1)'S(t)$, that is $\varphi(t)'\tilde{S}(t) = \varphi(t+1)'\tilde{S}(t)$. Summing up to $t = 2$ the relation $\tilde{S}_\varphi(t) = S_\varphi(0) + \tilde{G}_\varphi(t)$ we have:

$$\varphi(2)' \tilde{S}(2) = \varphi(1)' \tilde{S}(0) + \varphi(1)' (\tilde{S}(1) - \tilde{S}(0)) + \varphi(2)' (\tilde{S}(2) - \tilde{S}(1))\,.$$

Subtracting $\varphi(2)'\tilde{S}(2)$ on both sides gives $\varphi(2)'\tilde{S}(1) = \varphi(1)'\tilde{S}(1)$ which is $\varphi(t)'\tilde{S}(t) = \varphi(t+1)'\tilde{S}(t)$ for $t = 1$. Proceeding by induction we show $\varphi(t)'\tilde{S}(t) = \varphi(t+1)'\tilde{S}(t)$ for $t \in \{2, \ldots, T-1\}$ as required.

Exercise N° 11 [Discounted value process and equivalent martingale measure].

We have to prove that, for a given EMM $\mathbb{Q}$ (thus, we are under the AAO principle) and $\varphi \in \Phi$ any self-financing strategy, then the discounted value process $\tilde{S}_\varphi(t)$ is a $\mathbb{Q}$-martingale with respect to the filtration $\mathbb{F}$.

By the self-financing property of φ, we have $\tilde{S}_\varphi(t) = S_\varphi(0) + \tilde{G}_\varphi(t)$ for all $t \in \{0, \ldots, T\}$. This result implies:

$$\tilde{S}_\varphi(t+1) - \tilde{S}_\varphi(t) = \tilde{G}_\varphi(t+1) - \tilde{G}_\varphi(t) = \varphi(t+1)'(\tilde{S}(t+1) - \tilde{S}(t))\,.$$

Now, under the AAO $\tilde{S}(t)$ is a $\mathbb{Q}$-martingale and thus $E^{\mathbb{Q}}[\tilde{S}(t+1) - \tilde{S}(t)] = 0$. This result implies that $E^{\mathbb{Q}}[\tilde{S}_\varphi(t+1) - \tilde{S}_\varphi(t)] = 0$ and therefore $\tilde{S}_\varphi(t)$ is said to be a *martingale transform* of $\tilde{S}(t)$ by φ, being the sequence $\varphi(t+1)$ predictable given $\mathcal{F}_t$. This means, by the *Martingale Transform Lemma* [see Bingham and Kiesel (2004), Lemma 3.4.1], that $\tilde{S}_\varphi(t)$ is a $\mathbb{Q}$-martingale with respect to the filtration $\mathbb{F}$.