Fixed Income and Credit Risk

Lecture 3

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Fixed Income and Credit Risk

Lecture 3 - Part I

General Theories of Interest Rates

Outline of Lecture 3 - Part I

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3.1 General theories of interest rates

In this section we introduce four theories attempting to explain the term structure of interest rates (assuming ZCB prices are observed or estimated from CB prices). The first three are based upon general economic (useful) reasoning. The last theory, the **arbitrage-free pricing theory**, is the one we will follow during the course:

- *Expectations Hypothesis Theory*
- *Liquidity Preference Theory*
- *Market Segmentation Theory*
- *Arbitrage-Free Pricing Theory*

3.2 The Expectations Hypothesis Theory

- This is the most popular simple model of the term structure, and we distinguish between the *pure expectation hypothesis* (PEH) and the *expectation hypothesis* (EH).

- The PEH says that the expected excess returns on long-term over short-term bonds are zero.

- The EH allows expected excess returns on the long-term bond to depend on the maturity but not on time.

Different Forms of the Pure Expectations Hypothesis

☐ We distinguish different forms of the PEH, according to the time horizon over which expected excess returns are zero.

i) A first form of the PEH equates the one-period expected return of a bond with one-period to maturity with that of a bond with $(T-t)$ periods to maturity.

$\rightarrow$ The one-period return on a one-period bond is known in advance to be,

$$(1+Y(t,t+1)) = B(t+1,t+1)/B(t,t+1) = B(t,t+1)^{-1}$$

$\rightarrow$ The expected one-period return on a bond with $(T-t)$ periods to maturity is ($I_t =$ Investor's information):

$$E[(1+R_{t,t+1}^{T-t}) \mid I_t]$$

$\Rightarrow$ This form of PEH says therefore :

$$\begin{aligned}(1+Y(t,t+1)) &= E[(1+R_{t,t+1}^{T-t}) \mid I_t] \\ &= (1+Y(t,T))^{T-t} E_t[(1+Y(t+1,T))^{-(T-t-1)}]\end{aligned}$$

ii) A second form of the PEH equates the $(T-t)$-period return on an $(T-t)$-period bond to the expected return from rolling over one-period bonds for $(T-t)$ periods:

$$
\begin{aligned}
(1+Y(t,T))^{T-t} &= E_t[(1+Y(t,t+1))\dots(1+Y(T-1,T))] \\
\text{where } (1+Y(t,T))^{T-t} &= (B(T,T)/B(t,T)) = B(t,T)^{-1}
\end{aligned}
$$

iii) If that relation holds for all residual maturities $(T-t)$, once we consider short forward rates $(T=\tau)$, we can also write :

$$
(1+Y(t,\tau-1,\tau)) = \frac{(1+Y(t,\tau))^{\tau-t}}{(1+Y(t,\tau-1))^{\tau-1-t}} = E_t[(1+Y(\tau-1,\tau))]\,.
$$

Under this form of the PEH, the $(\tau-1-t)$-period-ahead short forward rate equals the expected $(\tau-1-t)$-period-ahead spot rate.

$iv)$ If the same relation holds again over all residual maturities $(T-t)$, we have:

$$\begin{aligned}
& (1+Y(t,T))^{T-t} \\
= \quad & (1+Y(t,t+1))E_t\left[(1+Y(t+1,t+2))\ldots(1+Y(T-1,T))\right] \\
= \quad & (1+Y(t,t+1))E_t\left[E_{t+1}\left[(1+Y(t+1,t+2))\ldots(1+Y(T-1,T))\right]\right] \\
= \quad & (1+Y(t,t+1))E_t\left[(1+Y(t+1,T))^{T-t-1}\right]
\end{aligned}$$

□ Observe that $iv)$ is incompatible with $i)$ whenever interest rates are random.

Indeed, $E(1/X) \neq 1/E(X)$.

Implications of the Log Pure Expectations Hypothesis

☐ Once the PEH is formulated in logs (continuously compounded yields), it is easy to state its implications for longer term bonds :

a) The one-period continuously compounded yield $R(t, t+1)$ (which is the same as the one-period geometric return on a one-period bond) should equal the expected geometric return on a $(T-t)$-period bond held for one period [see Lecture 1, slide 42]:

$$R(t, t+1) = E_t[\ln(B(t+1, T)/B(t, T))] = E_t[r^{T-t}_{t,t+1}]$$

$b)$ A long-term $(T-t)$-period yield $R(t,T)$ should equal the expected sum of $(T-t)$ successive one-period yields (i.e. returns) rolled over for $(T-t)$ periods:

$$R(t,T) = \frac{1}{T-t} \sum_{i=0}^{T-t-1} E_t[R(t+i, t+i+1)]$$

$c)$ the $(\tau-1-t)$-period ahead continuously compounded short forward rate should equal the expected one-period cont comp yield $(\tau-1-t)$ periods ahead:

$$R(t,\tau-1,\tau) = E_t[R(\tau-1,\tau)]$$

$\Rightarrow$ the cont comp short forward rate $R(t,\tau,\tau+1)$ should follow a martingale:

$$R(t,\tau,\tau+1) = E_t[R(\tau,\tau+1)] = E_t E_{t+1}[R(\tau,\tau+1)] = E_t[R(t+1,\tau,\tau+1)]$$

- ☐ If any of the equations at a), b) and c) hold for any residual maturity and date t, then the other equations also hold for all residual maturities and t.
- ☐ Also, if any of these equations hold for $(T - t) = 2$ at some date t, then the other equations also hold for $(T - t) = 2$ at the same date t.
- ☐ However, a), b) and c) are not generally equivalent for particular $(T - t)$ and t.

- The EH theory says that (using the cont comp notation) this difference:

$$R(t,T) - \frac{1}{T-t}\sum_{i=0}^{T-t-1} E_t[R(t+i,t+i+1)] =: TP(t,T)\,,$$

  called *Term Premia* of the yield $R(t,T)$, is constant over time and depends only on $T-t$: $TP(t,T) = TP(T-t)$. The PEH says that $TP(t,T) = 0$.

- $\frac{1}{T-t}\sum_{i=0}^{T-t-1} E_t[R(t+i,t+i+1)]$ is called *Expectation Component* of $R(t,T)$ and it is denoted $EX(t,T)$. This part of the yield provides information about **market expectations (forecast !) on future short rates** over an horizon given by the residual maturity $T-t$.

- It is related to the expected long-run level of the short rate.

- $EXS(t,T) = EX(t,T) - R(t,t+1)$ denotes the expectation component of the spread $S(t,T) = R(t,T) - R(t,t+1)$, and it gives information about expected long-run variation of the short rate with respect to its actual value.

- Thus, $TP(t,T)$ is the amount by which the long rate ($R(t,T)$) exceeds the expected return from investing in a series of short-term instruments.

- It is a **risk compensation** required to hold a long-term instead of a short-term debt.

3.3 Is the Expectations Hypothesis Empirically Verified ?

3.3.1 A Gaussian VAR-based answer

□ Let us calculate $TP(t,T)$ by means of a Gaussian VAR model giving us the possibility to determine $E_t[R(t+i,t+i+1)$, i.e. to forecast, at t, $R(t+i,t+i+1)$.

□ Our Gaussian VAR-distributed factor is $X_t = (r_t, S_t, g_t)'$ (1-Q short rate, 10-year spread, GDP growth). $S_t = R(t,t+40\,Q) - R(t,t+1\,Q)$, $g_t = \ln(GDP_t/GDP_{t-1})$.

□ To provide reliable forecasts we take information from both the bond market (short rate and spread) and the macro-economy (economic activity).

- If (X_t) is a Gaussian VAR(1) process, we assume:

$$X_{t+1} = \nu + \Phi X_t + \varepsilon_{t+1} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} + \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix} X_t + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix},$$

where ε_t is a 3-dimensional Gaussian white noise with $\mathcal{N}(0, \Omega)$ distribution.

- If (X_t) is a Gaussian VAR(3) process, we assume:

$$X_{t+1} = \nu + \Phi_1 X_t + \Phi_2 X_{t-1} + \Phi_3 X_{t-2} + \varepsilon_{t+1}.$$

- Let us calculate, at date t, the k-step ahead forecast (denoted $X^e_{t+k\,|\,t}$) with a VAR(1) model:

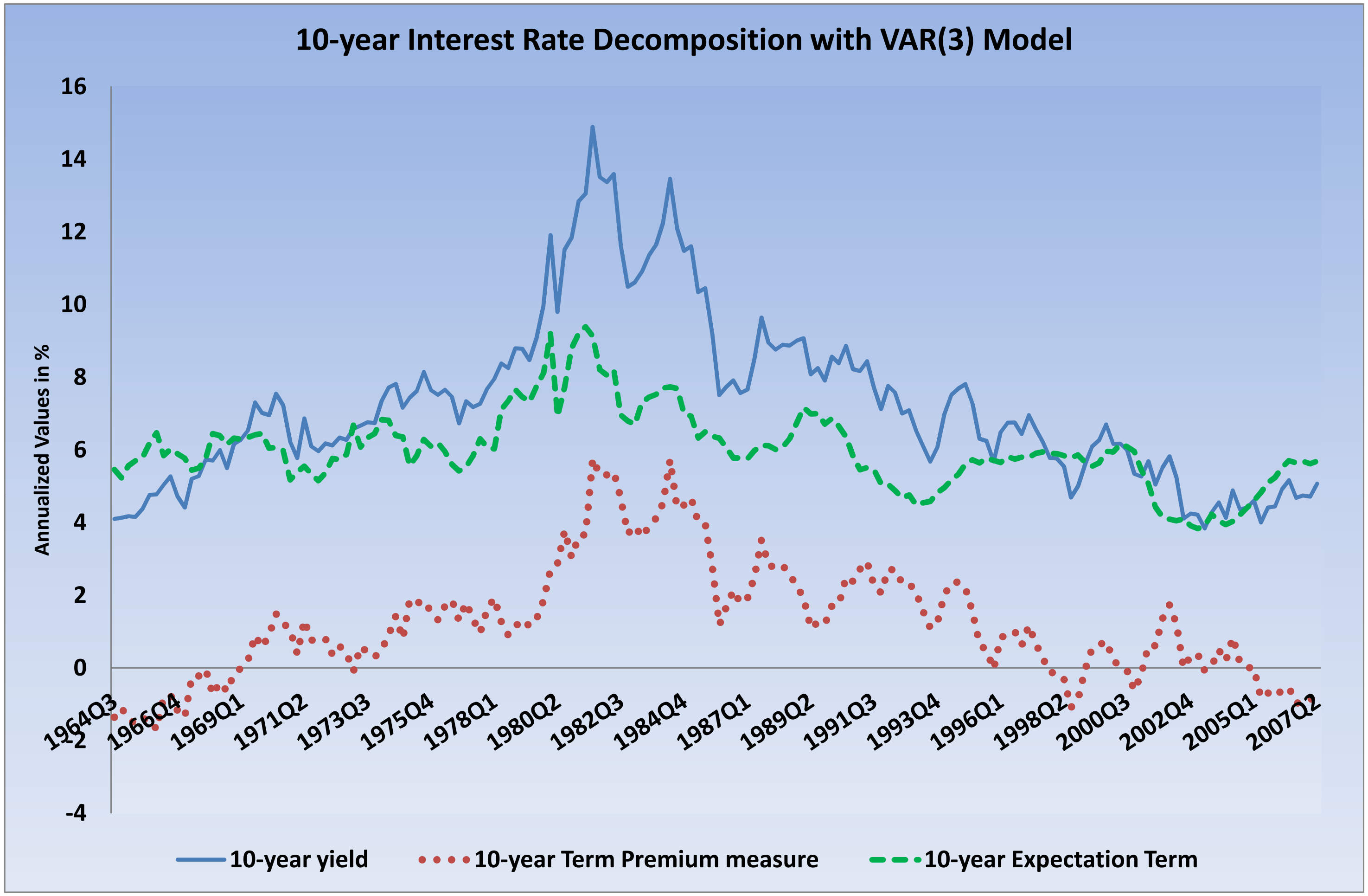
$$X^e_{t+k\,|\,t} := E_t[X_{t+k}] = (I_3 + \Phi + \ldots + \Phi^{k-1})\,\nu + \Phi^k X_t.$$

□ With a VAR(3) model, we recursively calculate the forecast:

$$X^e_{t+k\,|\,t} := E_t[X_{t+k}] = \nu + \Phi_1\, E_t[X_{t+k-1}] + \Phi_2\, E_t[X_{t+k-2}] + \Phi_3\, E_t[X_{t+k-3}]\,,$$

starting from $E_t[X_{t+1}] = \nu + \Phi_1\, X_t + \Phi_2\, X_{t-1} + \Phi_3\, X_{t-2}$.

□ Picture → Bold line: 10-year interest rate. Thin line: short rate. Dotted line: model-based 10-year term premium.


10-year Interest Rate Decomposition with VAR(3) Model
Annualized Values in %
16
14
12
10
8
6
4
2
0
-2
-4
1964Q3
1966Q4
1969Q1
1971Q2
1973Q3
1975Q4
1978Q1
1980Q2
1982Q3
1984Q4
1987Q1
1989Q2
1991Q3
1993Q4
1996Q1
1998Q2
2000Q3
2002Q4
2005Q1
2007Q2
10-year yield
10-year Term Premium measure
10-year Expectation Term

3.3.2 A regression-based answer

- What does the empirical evidence suggest about the Expectation Hypothesis ?

- Given the one-period geometric bond return $r_{t,t+1}^{(h)} = \ln \dfrac{B(t+1,T)}{B(t,T)}$, the one-period excess bond return is $ex_{t+1}^{(h)} = r_{t,t+1}^{(h)} - R(t,t+1)$ (with $h = T - t$).

- We can equivalently write:

$$ex_{t+1}^{(h)} = -[R(t+1,T) - R(t,T)]\,(T-t-1) + [R(t,T) - R(t,t+1)]$$

- and thus:

$$R(t+1,T) - R(t,T) = -\frac{1}{T-t-1}\,ex_{t+1}^{(h)} + \frac{1}{T-t-1}\,[R(t,T) - R(t,t+1)]$$

- The EH states $E_t(ex_{t+1}^{(h)}) = 0$, and to test for it we can run the following regression:

$$R(t+1,T) - R(t,T) = \phi_0(h,1) + \phi_1(h,1)\,\frac{1}{T-t-1}\,[R(t,T) - R(t,t+1)] + u_{t+1}(h,1)$$

- It is also possible to prove that, given an m-period bond return $(m < T-t)$ $r_{t,t+m}^{(h)} = \frac{1}{m}\ln\frac{B(t+m,T)}{B(t,T)}$, with $ex_{t+m}^{(h)} = r_{t,t+m}^{(h)} - R(t,t+m)$:

$$R(t+m,T) - R(t,T) = \phi_0(h,m) + \phi_1(h,m)\,\frac{m}{T-t-m}\,[R(t,T) - R(t,t+m)] + u_{t+m}^{(h,m)}$$

- These regressions are called in the literature **Campbell-Shiller regressions**, from the paper by Campbell and Shiller (1991, RES).

- If we focus on the first regression ($m = 1$), testing the EH means testing that $\phi_0(h, 1) = 0$ and $\phi_1(h, 1) = 1$. A well known empirical feature of the U.S. yield curves is that $\widehat{\phi_1}(h, 1) < 0$, for all residual maturities and (in general) $|\widehat{\phi_1}(h, 1)|$ increasing as far as h increases.

- This is called the **violation of the EH Theory** or **EH Puzzle**.

- Let us see some empirical result about $\widehat{\phi_1}(h, 1)$. We take the GSW (2007) data base with quarterly observations from 1964 : Q1 to 2007 : Q2.

h	3-Q	4-Q	8-Q	12-Q	20-Q	40-Q
$\widehat{\phi_1}(h,1)$	-0.49	-0.74	-0.98	-1.20	-1.55	-2.57
st. dev.	[0.28]	[0.40]	[0.68]	[0.82]	[0.93]	[1.19]

- Let us see an application to CRSP data set on the U. S. term structure of interest rates.

- The sample covers the period from June 1964 to December 1995.

- We have 379 monthly observations for each of the nine maturities : 1, 3, 6 and 9 months and 1, 2, 3, 4 and 5 years.

- How do $\widehat{\phi_1}(h, m)$ behaves ?

Short Horizon	$m = 3$ months	$m = 6$ months	$m = 9$ months
$h = 6$ months	-0.6942 (0.2533)		
$h = 9$ months	-0.8863 (0.3238)	-0.4023 (0.2429)	
$h = 12$ months	-1.3226 (0.3530)	-0.7867 (0.2381)	-0.4371 (0.1312)

Long Horizon	$m = 1$ year	$m = 2$ years	$m = 3$ years
$h = 4$ years	-1.8078 (0.2981)	-0.8380 (0.2889)	-0.0421 (0.2682)
$h = 5$ years	-1.7470 (0.3291)	-0.9720 (0.3199)	-0.2378 (0.3283)

3.3.3 Cochrane and Piazzesi (2005)

☐ Cochrane and Piazzesi (2005) generalize the previous approach and consider the following regressions:

$$ex_{t+1}^{(h)} = \beta_0(h) + \beta_1(h)\, R(t, t+1) + \sum_{j=2}^{5} \beta_j(h)\, f_t^{(j)} + \varepsilon_{t+1}^{(h)}, \;\; h \in \{1, \ldots, 5\}$$

$$f_t^{(j)} = R(t, t+j-1, t+j) = \ln(B(t, t+j-1)/B(t, t+j))\,.$$

☐ They document a "tent shape" for the estimates of the coefficients $(\beta_j(h))_{j=1}^{5}$, for $h \in \{1, \ldots, 5\}$, t is measured in years and, thus, returns are calculated on a yearly basis.

- They also document that this "tent shape" factor is not fully captured by level, slope and curvature factors.

- It reflects a four- to five-year spread that is ignored by factor models. Indeed, its variance is most explained by the 2^{nd} PC (58.7%), that is the SLOPE factor used by Campbell and Shiller (1991),...

- ...but it is also explained by the 4^{th} PC (24.3%). It loads heavily on the four- to five-year yield spread.

3.4 The Liquidity Preference Theory

- Investors usually prefer short-term investments to long-term ones (they do not like to tie their capital up for too long).

- An explanation of that theory could be the following : prices of longer-term bonds tend to be more volatile (riskier!) than short-term bonds. Thus, investors will only invest in more volatile securities if they have higher expected return (often named *risk premium*) to offset the higher risk.

- This theory implies therefore that $TP(t,T) \uparrow$ as $T \uparrow$ when I move from a short-term bond investment to a long-term one.

3.5 The Market Segmentation Theory

- ☐ Each investor has in mind, on the basis of his system of preferences, an appropriate set of bonds and maturity dates that are adapted to their investment plans.

- ☐ For example, life insurance companies require long-term bonds to match their long-term liabilities. In contrast, banks are likely to prefer short-term bonds to reflect the needs of their customers.

- ☐ Different group of investors, with different system of preferences and different attitudes toward risk, act in different ways.

- The *basic form* of the market segmentation theory says that there is no reason why there should be any interaction between different groups. This means that bond prices in different maturity bands will change in unrelated ways.

- *More realistically*, investor who prefer certain maturities (short/medium/long) may shift their investments if they think that bonds in a different maturity band are particularly cheap.

- What is implicit in that theory, and in contrast with the liquidity preference one, the term premium $TP(t,T)$ is no more obliged to be an increasing function of the time-to- maturity. Long-term investors will ask for risk premium compensation in order to move their investment toward the short-term band of the yield curve.

3.6 The Arbitrage-Free Pricing Theory

□ The remainder of this course consider the pricing of bonds in a market which is arbitrage-free. This theory pulls together, in a mathematically precise way, the expectation, liquidity preference and market segmentation theory.

□ It is based on the Absence of Arbitrage Opportunity (A.A.O.) principle. The *no-arbitrage bond price* is the one satisfying the A.A.O. The *fair price* is the no-arbitrage one.

□ $R(t,T) = \frac{1}{T-t}\sum_{i=0}^{T-t-1} E_t[R(t+i,t+i+1)] + TP(t,T)$ where $TP(t,T)$ is a time-varying (model-dependent) function not necessarily increasing with the time-to-maturity.

□ In that course we do not consider the *General Equilibrium* approach as a principle to specify the fair price. $G.E. \Rightarrow A.A.O.$ but $A.A.O.$-based asset pricing models are more flexible (closer to the data) that $G.E.$-based asset pricing models.

Fixed Income and Credit Risk

Lecture 3 - Part II

No-Arbitrage Asset Pricing Theory

in Discrete Time

Outline of Lecture 3 - Part II

3.6 No-arbitrage asset pricing theory in a one-period model

3.6.1 Outline of the main results

3.6.2 The setup of the one-period model

3.6.3 Complete and incomplete markets

3.6.4 Linear pricing: the law of one price

3.6.5 Positive pricing: the absence of arbitrage opportunity principle

3.6.6 Risk-free payoffs and risk-neutral probabilities

3.6.7 Stochastic discount factors

3.6.8 Stochastic discount factor and change of probability measure

3.7 Arbitrage theory in a dynamic discrete-time model

3.7.1 The setup of the discrete-time model

3.7.2 Self-financing trading strategies

3.7.3 The no-arbitrage condition

3.7.4 Attainable payoffs

3.7.5 Complete markets: uniqueness of the EMM

3.7.6 Stochastic discount factors

3.7.7 Stochastic discount factor and change of probability measure

3.6 No-arbitrage asset pricing theory in a one-period model

3.6.1 Outline of the main results

- □ In this section we will study the mathematical structure of a **simple one-period model** (i.e., two dates!) of a financial market.

- □ We will consider a **finite number of assets** (bonds, stocks, commodities, currencies). Their initial prices at time $t = 0$ (today) are known, their future prices at time $t = T$ are described as random variables on some probability space.

- □ Trading takes place at time $t = 0$, and $t = T$ is the terminal date for all economic activities.

- We will assume (for ease of exposition) that all random variables take a finite number of possible values (finite state space).

- The price of any asset is considered "fair" if it satisfies the **absence of arbitrage opportunity principle**: there are no trading opportunities in the market which yield a risk-free profit.

- The absence of such arbitrage opportunities is characterized by (identified with) the **existence** of an **equivalent martingale measure (EMM)** or a **positive stochastic discount factor** (SDF).

- Under such a measure, discounted (from $t = T$ to $t = 0$) asset prices are **martingales**.

- Under the no-arbitrage principle, the **EMM (i.e., the SDF) is unique** if and only if the **financial market is complete**.

- If the financial market is **incomplete**, under the no-arbitrage principle there exists an **infinity of EMM such that** discounted asset prices are martingales.

3.6.2 The setup of the one-period model

- The frictionless financial market contains $d+1$ traded basic assets, whose prices at time $t = 0$ are denoted by the vector $S(0) \in \mathbb{R}_+^{d+1}$: $S(0) = [S_0(0), S_1(0), \ldots, S_d(0)]'$.

- At time T, the owner of the financial asset number i receives a random payment (the asset value at T, the asset payoff) depending on the state of the world.

- We introduce a **finite** probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a finite number $|\Omega| = N$ of points (states of the world) $\omega_1, \ldots, \omega_j, \ldots, \omega_N$ each with positive probability $p_j = \mathbb{P}(\{\omega_j\}) > 0$. This means that every state of the world is possible.

- $\mathcal{F}$ is the set of subsets of Ω (the events that can happen in the world) on which $\mathbb{P}(.)$ is defined (we can quantify how probable these events are).

- We can now write the random payment (discrete random variable) arising from the financial asset i as the N-dimensional vector

$$S_i(T) = [S_i(T, \omega_1), \ldots, S_i(T, \omega_j), \ldots, S_i(T, \omega_N)]' \in \mathbb{R}_+^N.$$

- At time $t = 0$ agents can buy and sell financial assets. The portfolio position of an individual agent is given by a **trading strategy** $\varphi \in \mathbb{R}^{d+1}$, that is a $(d+1)-$dimensional vector $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)'$. Here φ_i denotes the quantity of the i^{th} asset bought at time $t = 0$, which may be negative as well as positive (we allow short positions).

☐ The dynamics of our model using the trading strategy φ are as follows:

– at time $t = 0$ we invest the amount $S_\varphi(0) := S(0)'\varphi = \sum_{i=0}^{d} \varphi_i S_i(0)$

– and at time $t = T$ we receive the random payment $S(T,\omega)'\varphi = \sum_{i=0}^{d} S_i(T,\omega)\varphi_i$

depending of the realized state ω of the world.

☐ Let us represent all the possible payoffs of the $d+1$ assets in the following $[(d+1) \times N]$-matrix $\mathbf{S}$:

$$\mathbf{S} = \begin{bmatrix} S_0(T,\omega_1) & \dots & S_0(T,\omega_N) \\ S_1(T,\omega_1) & \dots & S_1(T,\omega_N) \\ \vdots & \ddots & \vdots \\ S_d(T,\omega_1) & \dots & S_d(T,\omega_N) \end{bmatrix}$$

☐ The payoff matrix $\mathbf{S}$ has inverse iff it is square $((d+1) = N)$ and of full rank.

- Neither of these properties is assumed to be true in general. However, even if $\mathbf{S}$ is not square, it may have **left inverse** or **right inverse**.

- **Left Inverse :** it is a $[N \times (d+1)]$-matrix $L^{(S)}$ such that $L^{(S)}\,\mathbf{S} = I_N$. $L^{(S)}$ exists iff $rank(\mathbf{S}) = N$, which occurs if $(d+1) \geq N$ and the $cols(\mathbf{S})$ are linearly independent. We have $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\,\mathbf{S}'$ which exists iff $(\mathbf{S}'\mathbf{S})$ is invertible.

- **Right Inverse :** it is a $[N \times (d+1)]$-matrix $R^{(S)}$ such that $\mathbf{S}\,R^{(S)} = I_{d+1}$. $R^{(S)}$ exists iff $rank(\mathbf{S}) = d+1$, which occurs if $(d+1) \leq N$ and $rows(\mathbf{S})$ are linearly independent. We have $R^{(S)} = \mathbf{S}'\,(\mathbf{S}\mathbf{S}')^{-1}$ which exists iff $(\mathbf{S}\mathbf{S}')$ is invertible.

- We have $(L^{(S)})' = R^{(S')}$ being $(\mathbf{S}\mathbf{S}')$ and $(\mathbf{S}'\mathbf{S})$ symmetric.

- ☐ The N-dimensional vector of possible payments at $t = T$ from the trading strategy φ (i.e. the random payoff of the portfolio φ) is : $S_{\varphi}(T) = \mathbf{S}'\varphi$.

- ☐ We define the **asset span** $\mathcal{M}(\mathbf{S})$ as the set of all possible payoffs that can be **generated** (**replicated**) by trading (via φ) the $(d+1)$ basic assets. More formally: $\mathcal{M}(\mathbf{S}) = \{y \in \mathbb{R}^N \,:\, y = \mathbf{S}'\varphi\,, \varphi \in \mathbb{R}^{d+1}\} \subseteq \mathbb{R}^N$.

- ☐ $\mathcal{M}(\mathbf{S})$ is a linear (vector) space : its dimension is given by the number k of linearly independent payoffs $S_i(T) \Rightarrow k = rank(\mathbf{S}') =$ number of linearly independent rows of $\mathbf{S}$ (thus, $k \leq N$).

□ The $k \leq (d+1)$ basic assets with linearly independent payoffs are called **non-redundant** assets. The remaining $(d+1-k)$ assets are called **redundant** (their payoffs can be replicated by linear combination of the non-redundant ones).

□ We define the **market span** $\mathcal{M}(S(0), \mathbf{S})$ as:

$$\mathcal{M}(S(0), \mathbf{S}) = \left\{(x, y')' , x \in \mathbb{R} , y \in \mathbb{R}^N \, : \, x = -S(0)'\varphi , y = \mathbf{S}'\varphi , \varphi \in \mathbb{R}^{d+1}\right\} \subseteq \mathbb{R}^{N+1}.$$

It collects all today's and tomorrow's cash flows that can be achieved by trading the basic assets. It captures the set of allocations of purchasing power through time and states that can be achieved by some portfolio, and among which the agents choose the best one.

3.6.3 Complete and incomplete markets

- The financial market $\{S(0), \mathbf{S}\}$ is said to be **complete** if, for any possible payoff $y \in \mathbb{R}^N$, there exists a (non necessarily unique) trading strategy $\varphi \in \mathbb{R}^{d+1}$ such that $y = \mathbf{S}'\varphi$.

- We have that :

  – the financial market $\{S(0), \mathbf{S}\}$ is complete iff $k = rank(\mathbf{S}') = N$ ($\mathcal{M}(\mathbf{S}) = \mathbb{R}^N$);

  – if $k = rank(\mathbf{S}') < N$, then $\{S(0), \mathbf{S}\}$ is said to be **incomplete** ($\mathcal{M}(\mathbf{S}) \subset \mathbb{R}^N$);

  – a *necessary but not sufficient* condition for $\{S(0), \mathbf{S}\}$ to be complete is : $(d + 1) \geq N$.

- ☐ Given a complete market $\{S(0), \mathbf{S}\}$, the *replicating strategy* φ, for a given payoff $y \in \mathbb{R}^N$, *is unique* iff $N = k = d + 1$. We have : $\varphi = (\mathbf{S}')^{-1}y$ for any $y \in \mathbb{R}^N$. It is a **complete market without redundant assets**.

- ☐ For any $y = e_j = (0, \ldots, 0, 1, 0, \ldots, 0)'$, $j \in \{1, \ldots, N\}$ (the j^{th} element of the canonical basis in $\mathbb{R}^N$), the associated (**unique**) portfolio $\varphi^{ad,j} = (\mathbf{S}')^{-1}e_j$ replicates the so called **state-j Arrow-Debreu security** providing the **state claim** (state-j payoff) e_j.

- ☐ In that case the j^{th} column of $(\mathbf{S}')^{-1}$, $\widetilde{S}_j$ (say), contains the unique portfolio weights $\varphi^{ad,j} = (\mathbf{S}')^{-1}e_j := \widetilde{S}_j$ that replicates the state-j A-D security.

- Thus, **completeness of the market without redundant assets** means that there **exists a unique replicating portfolio** for each A-D security.

- Moreover, given that any payoff $y \in \mathbb{R}^N$ (we would like to replicate) can be written as a linear combination of A-D security payoffs, that is $y = \sum_{j=1}^{N} y_j \times e_j$, we also have that:

$$\varphi = (\mathbf{S}')^{-1} \left(\sum_{j=1}^{N} y_j \times e_j \right) = \sum_{j=1}^{N} y_j \times \varphi^{ad,j},$$

that is the replicating portfolio φ is a linear combination of the N portfolios that replicate the N A-D securities. Each $\varphi^{ad,j}$ is weighted by the payoff value y_j associated to the state ω_j.

☐ Given a complete market $\{S(0), \mathbf{S}\}$, the **replicating strategy φ is not unique** when $N = k < d + 1$ (exercise!). It is a **complete market with** $(d + 1 - N)$ **redundant assets**. This also means that the portfolio $\varphi^{ad,j}$, replicating the payoff of the state-j A-D security, always exists but is not unique (for any $\omega_j \in \Omega$).

☐ Let us consider an **incomplete market**:

$i)$ **without redundant assets** $(k = d + 1 < N)$, any $y \in \mathcal{M}(\mathbf{S})$ has a unique replicating portfolio φ (exercise!);

$ii)$ **with redundant assets** $(k < d+1,\ k < N)$, any $y \in \mathcal{M}(\mathbf{S})$ has infinitely many replicating portfolios φ (exercise!).

☐ If the (non square) payoff matrix $\mathbf{S}$ has **only left inverse** $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\,\mathbf{S}'$:

- $N < (d+1)$ and N linearly independent rows and columns in $\mathbf{S}$;
- the market is complete and we have $(d+1-N)$ redundant assets.

☐ If the (non square) payoff matrix $\mathbf{S}$ has **only right inverse** $R^{(S)} = \mathbf{S}'\,(\mathbf{S}\mathbf{S}')^{-1}$:

- we have $N > (d+1)$ and $(d+1)$ linearly independent rows and columns in $\mathbf{S}$;
- this means that the market is incomplete and no security is redundant.

□ The payoff matrix $\mathbf{S}$ has **both left inverse and right inverse** iff $\mathbf{S}$ is a **square nonsingular matrix**. This means that, if $\mathbf{S}$ admits both $L^{(S)}$ and $R^{(S)}$, then we have a complete market without redundant assets.

□ **Theorem :** The financial market $\{S(0), \mathbf{S}\}$ is complete if and only if $\mathbf{S}$ admits left inverse, namely $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\,\mathbf{S}'$.

3.6.4 Linear pricing : the law of one price

- **Law of One Price :** two assets with the same payoff vector have the same price. More generally, all portfolios with the same payoff have the same price. *Formally* : if $\mathbf{S}'\varphi^* = \mathbf{S}'\varphi^{**}$, with $\varphi^* \neq \varphi^{**}$ and $\varphi^*, \varphi^{**} \in \mathbb{R}^{d+1}$, then $S_{\varphi^*}(0) = S_{\varphi^{**}}(0)$. *Equivalently* : the LOP holds iff every portfolio $\varphi \in \mathbb{R}^{d+1}$ s.t. $\mathbf{S}'\varphi = 0$ has $S_\varphi(0) = 0$.

- This is the first economic principle we introduce in order to define what a reasonable (fair) price is for an asset or a portfolio. We are going to see that the LOP leads the asset price function (of the payoff) to be a **linear** function of the payoff.

☐ **The Payoff Pricing Function :** for any vector of basic asset prices $S(0)$, we define a mapping $q : \mathcal{M}(\mathbf{S}) \mapsto \mathbb{R}$ that assigns to each payoff in the asset span the price of the portfolio that generate that payoff. Formally:

$$q(y) = \{S \in \mathbb{R} \,:\, S = S_\varphi \text{ for some } \varphi \in \mathbb{R}^{d+1} \text{ such that } y = \mathbf{S}'\varphi\}\,.$$

☐ In general the mapping $q(.)$ is a correspondence rather than a single-valued function. If LOP holds, then $q(.)$ is a single-valued and linear.

☐ **Theorem :** The Law of One Price holds if and only if $q(.)$ is a linear functional of the asset span $\mathcal{M}(\mathbf{S})$.

[Proof : exercise.]

□ Under the LOP, $q(.)$ is single-valued and linear and therefore the price $S_\varphi(0)$ of the portfolio φ is given by $S_\varphi(0) = S(0)'\varphi$. Indeed :

$$
\begin{aligned}
S_\varphi(0) &= q\left(\mathbf{S}'\varphi\right) = q\left(\sum_{i=0}^{d} S_i(T)\,\varphi_i\right) \\
&= \sum_{i=0}^{d} \varphi_i\, q(S_i(T)) \\
&= \sum_{i=0}^{d} \varphi_i\, S_i(0) \\
&= S(0)'\varphi\,.
\end{aligned}
$$

☐ Under the LOP, even in the case of **incompleteness** of $\{S(0), \mathbf{S}\}$, if there are **no redundant securities** ($k = rank(\mathbf{S}') = d + 1 < N$), then $R^{(S)}$ is well defined and any payoff $y \in \mathcal{M}(\mathbf{S})$ can be priced by the following formula: $q(y) = y' \, R^{(S)} \, S(0) = S(0)' \, L^{(S')} \, y$ [Proof: exercise].

☐ Moreover, let us consider an **incomplete market with redundant securities** ($k < d + 1$, $k < N$). Let us denote with $\bar{\mathbf{S}}$ the (k, N) payoff matrix of the no redundant assets, with $R^{(\bar{S})}$ its right inverse, and with $\bar{S}(0)$ the vector of these k asset prices. Then, any payoff $y \in \mathcal{M}(\mathbf{S})$ can be priced by the following formula: $q(y) = y' \, R^{(\bar{S})} \, \bar{S}(0) = \bar{S}(0)' \, L^{(\bar{S}')} \, y$ [Proof: exercise].

a) $q(.)$ is one of three operators that are related in a triangular fashion:

$$q: \qquad \mathcal{M}(\mathbf{S}) \longrightarrow \mathbb{R}\,; \;\; y = \mathbf{S}'\varphi \xrightarrow{q} S_\varphi\,.$$

b) the set of all possibles portfolios $\varphi \in \mathbb{R}^{d+1}$ is termed the *portfolio space*. The **portfolio pricing function** is a linear function assigning to each portfolio φ the associated price $S_\varphi(0) = S(0)'\varphi$:

$$S_\varphi(0): \qquad \mathbb{R}^{d+1} \longrightarrow \mathbb{R}\,; \;\; \varphi \xrightarrow{S_\varphi(0)} S(0)'\varphi\,.$$

c) the payoff matrix $\mathbf{S}$ can be interpreted as a linear operator (the **payoff operator**) from the portfolio space $\mathbb{R}^{d+1}$ to the asset span $\mathcal{M}(\mathbf{S})$:

$$\mathbf{S}: \qquad \mathbb{R}^{d+1} \longrightarrow \mathcal{M}(\mathbf{S})\,; \;\; \varphi \xrightarrow{\mathbf{S}} y = \mathbf{S}'\varphi\,.$$

☐ When the LOP holds, the portfolio pricing function $S_\varphi(0)$ can be represented as $S_\varphi(0) = q \circ \mathbf{S} \circ \varphi = q((\mathbf{S}(\varphi)) = q(\mathbf{S}'\varphi) = S(0)'\varphi$, that is it can be decomposed as:

$$S_\varphi(0) = q \circ \mathbf{S} \circ \varphi \,:\, \mathbb{R}^{d+1} \longrightarrow \mathcal{M}(\mathbf{S}) \longrightarrow \mathbb{R}$$

$$\varphi \xrightarrow{\mathbf{S}} y = \mathbf{S}'\varphi \xrightarrow{q} S(0)'\varphi\,.$$

☐ If **markets are complete** and if **the LOP holds**, then the payoff pricing function assigns a **unique price** to each state claim $e_j \in \{e_1, \ldots, e_N\}$. Let us denote the price of the state-j A-D security by $q_j^{(ad)} = q(e_j)$. We call $q_j^{(ad)}$ the **state price** of state j. Let us denote by $q^{(ad)} = [q_1^{(ad)}, \ldots, q_N^{(ad)}]'$ the **vector of state prices**.

☐ In that case we have the following results:

a) given that any payoff $y \in \mathcal{M}(\mathbf{S})$ can be written as $y = \sum_{j=1}^{N} y_j \times e_j$, we have that:

$$q(y) = \sum_{j=1}^{N} y_j \times q(e_j) = \sum_{j=1}^{N} y_j \times q_j^{(ad)} \, ;$$

b) this means that the price $S_i(0)$ of any basic asset $i \in \{1, \ldots, N\}$ can be written as:

$$S_i(0) = q[S_i(T)] = q\left[\sum_{j=1}^{N} S_i(T, \omega_j) \times e_j\right] = \sum_{j=1}^{N} S_i(T, \omega_j) \times q_j^{(ad)} \, ;$$

in matrix notation $S(0) = \mathbf{S}\, q^{(ad)}$;

c) we have $q^{(ad)} = L^{(S)}\, S(0)$: we are able to calculate the vector of state prices, giving the possibility to price any payoff $y \in \mathbb{R}^N \equiv \mathcal{M}(\mathbf{S})$. If the market is not complete it could be $e_j \notin \mathcal{M}(\mathbf{S})$ and therefore $q(e_j)$ is not well defined.

□ **Theorem :** In the financial market $\{S(0), \mathbf{S}\}$ the LOP holds if and only if there is a payoff $q^* \in \mathcal{M}(\mathbf{S})$ such that:

$$q(y) = y' q^* \,, \;\; \forall \, y \, \in \, \mathcal{M}(\mathbf{S}) \,.$$

- This payoff $q^* \in \mathcal{M}(\mathbf{S})$ is unique (but not positive in general).
- If $\{S(0), \mathbf{S}\}$ is complete, then $q^* \equiv q^{(ad)} = L^{(S)}\, S(0)$.
- If $\{S(0), \mathbf{S}\}$ is incomplete without redundant assets, then $q^* = R^{(S)}\, S(0)$.
- If $\{S(0), \mathbf{S}\}$ is incomplete with redundant assets, then $\bar{q}^* = R^{(\bar{S})}\, \bar{S}(0)$.

3.6.5 Positive pricing : the absence of arbitrage opportunity principle

☐ We have seen that the LOP makes $q(.)$ a linear function on the asset span $\mathcal{M}(\mathbf{S})$. Now we are going to impose, to $S(0)$ and $\mathbf{S}$, to satisfy the Absence of Arbitrage Opportunity (AAO) principle. This is a stronger principle than LOP.

☐ An **arbitrage** in the financial market $\{S(0), \mathbf{S}\}$ is a trading strategy (a portfolio) $\varphi \in \mathbb{R}^{d+1}$ satisfying one of the following two conditions:

i) $S_\varphi(0) < 0$ and $S_\varphi(T) \geq 0$, i.e. $S(T,\omega)'\varphi \geq 0 \ \forall \omega \in \Omega$;

ii) $S_\varphi(0) = 0$ and $S_\varphi(T) \geq 0$, and there exists at least one state $\omega_j \in \Omega$ such that $S_\varphi(T,\omega_j) > 0$ ($S(T,\omega)'\varphi \geq 0 \ \forall \omega \in \Omega$ and $\exists\, \omega_j \in \Omega$ such that $S(T,\omega_j)'\varphi > 0$).

- Meaning of $i)$: we borrow money at $t = 0$ and we do not have to repay anything at $t = T$. Meaning of $ii)$: we potentially obtain wealth without any initial capital.

- The are **no arbitrage opportunities** in the financial market $\{S(0), \mathbf{S}\}$ when there is no arbitrage. That is to say, the following conditions must hold:

  $a)$ $S_\varphi(T) = 0$ implies $S_\varphi(0) = 0$ (the portfolio with a zero payoff has zero value);

  $b)$ $S_\varphi(T) \geq 0$, $S_\varphi(T) \neq 0$ implies $S_\varphi(0) > 0$ (the portfolio with positive payoff has a positive value).

- AAO $\Rightarrow$ LOP (from $a)$). AO exist if LOP does not holds.

- If $\{S(0), \mathbf{S}\}$ satisfies the AAO principle, then $q(.)$ is **linear and positive**: for any $y > 0$ we have $q(y) > 0$.

- This means that, if markets are complete and AAO is satisfied ($\Rightarrow$ LOP holds), then $q(e_j) = q_j^{ad} > 0$ for all $j \in \{1, \ldots, N\}$.

- Nevertheless, if the market is incomplete, we are not able to price any $y \in \mathbb{R}^N$. In particular, we can't price payoffs $y \in \mathbb{R}^N / \mathcal{M}(\mathbf{S})$. **How can I solve this problem?**

- Under the AAO principle, there exists a **linear positive Valuation Function** $\mathcal{Q}(y)$ which is an extension of $q(.)$ to the entire contingent claim space $\mathbb{R}^N$. We can price any payoff, satisfying at the same time the AAO principle.

☐ Formally:

$$\mathcal{Q}: \qquad \mathbb{R}^N \longrightarrow \mathbb{R}\,; \;\; y \xrightarrow{\mathcal{Q}} \mathcal{Q}(y)\,,$$

$$\mathcal{Q}(y) \;\;=\;\; q(y) \text{ for every } y \in \mathcal{M}(\mathbf{S})\,.$$

☐ The financial market $\{S(0), \mathbf{S}\}$ excludes arbitrage iff there **exists a (not unique in general) linear positive** valuation function $\mathcal{Q}(y)$.

$i)$ Thus, under AAO and given $y = \sum_{j=1}^{N} y_j \times e_j \in \mathbb{R}^N$, we have that:

$$\mathcal{Q}(y) = \sum_{j=1}^{N} y_j \times \mathcal{Q}(e_j) = \sum_{j=1}^{N} y_j \times q_j^{(ad)} = q^{(ad)\prime} y\,, \text{ with } \mathcal{Q}(e_j) = q_j^{(ad)} > 0$$

$ii)$ we can price **any** payoff $y \in \mathbb{R}^N$ **without determining** a portfolio φ such that $y = \mathbf{S}'\varphi$.

iii) Given that $\mathcal{Q}(.)$ is not unique in general (if the market is incomplete), then state price vector $q^{(ad)} > 0$ is not unique either (and, therefore, also the asset price $\mathcal{Q}(y)$).

iv) The asset price $\mathcal{Q}(y)$ is **independent of** the positive state price vector $q^{(ad)}$ iff $y \in \mathcal{M}(\mathbf{S})$ [see slides 68 and 76].

☐ Let us suppose that the financial market $\{S(0), \mathbf{S}\}$ excludes arbitrage. Then the financial market $\{S(0), \mathbf{S}\}$ is complete iff there exists a **unique linear positive** valuation function $\mathcal{Q}(y)$.

☐ Let us present that result in terms of $\{S(0), \mathbf{S}\}$!

☐ **First Fundamental Theorem of Asset Pricing :** In the financial market $\{S(0), \mathbf{S}\}$ there are no arbitrage opportunities if and only if there exists a strictly positive vector of state prices $q^{(ad)} \in \mathbb{R}^N_{++}$ such that:

$$S(0) = \mathbf{S}\, q^{(ad)}\,, \text{ with } q_j^{ad} = q(e_j)\,, e_j \in \{e_1, \ldots, e_N\}$$

$$q(e_j) \text{ price of state-}j \text{ Arrow-Debreu security}\,, \; e_j = (0, \ldots, \underbrace{1}_{j^{th} \text{ position}}, \ldots, 0)'$$

☐ **Second Fundamental Theorem of Asset Pricing :** Let us assume that the financial market $\{S(0), \mathbf{S}\}$ admits no arbitrage. There exists a unique strictly positive state price vector $q^{(ad)} \in \mathbb{R}^N_{++}$ if and only if the market is complete.

□ If the **market is incomplete**, under AAO, there exists several $q^{(ad)} > 0$ such that $S(0) = \mathbf{S}\, q^{(ad)}$. Nevertheless, that system of equations admits (in general) also non-positive solutions $q^{(ad)} \in \mathbb{R}^N$. These solutions do not guarantee to satisfy the AAO principle for any $y \in \mathbb{R}^N$.

□ **Theorem :** There exists a **positive valuation function** $Q(.)$ if and only if there exists a $q^{(ad)} \in \mathbb{R}^N_{++}$ solution of $S(0) = \mathbf{S}\, q^{(ad)}$. Each positive solution $q^{(ad)}$ defines a positive valuation function $Q(.)$ satisfying $Q(y) = q^{(ad)\prime} y$ for all $y \in \mathbb{R}^N$.

□ **Theorem :** Let us assume that the financial market $\{S(0), \mathbf{S}\}$ admits no arbitrage. Then, the market is complete if and only if the matrix equation $\mathbf{S}'\varphi = y$ has a solution $\varphi \in \mathbb{R}^{d+1}$ for any vector $y \in \mathbb{R}^N$.

□ Intuitively : $\{S(0), \mathbf{S}\}$ admits no arbitrage whenever the price vector $S(0)$ lies in the convex cone generated by the columns of $\mathbf{S}$, i.e. the vectors $S(T,\omega_j) := (S_0(T,\omega_j),\ldots,S_{d+1}(T,\omega_j))'$ with $j \in \{1,\ldots,N\}$:

$$S(0) \in \mathcal{K} = \left\{ k \in \mathbb{R}^N \;:\; k = \sum_{j=1}^{N} \lambda_j \, S(T,\omega_j) \,;\, \lambda_j > 0 \;\forall 1 \le j \le N \right\}.$$

□ Indeed : $S(0) = \mathbf{S}\, q^{(ad)} = \sum_{j=1}^{N} q_j^{(ad)} \, S(T,\omega_j)$

□ Proof of the First and Second FTAP $\rightarrow$ exercise!

☐ `Example 1:` Let us consider a bond market over a one-period only. This means that we consider only two dates: $t = 0$ and $t = 1$. At the date $t = 0$ two zero-coupon bonds ($i \in \{0, 1\}$, say) are available in the market:

- the first one (the asset $i = 0$) mature at $t = 1$ and has a price $B_0(0, 1) = 0.9$;
- the second one (the asset $i = 1$) mature at $t = 2$ and has a price $B_1(0, 2) = 0.81$;

- At date $t = 1$ we have $N = 3$ possible states of the world : ω_1, ω_2 and ω_3.

- The payoff matrix of the 2 assets, at date $t = 1$, is the following (2×3)-matrix $\mathbf{B}$ (say):

$$\mathbf{B} = \begin{bmatrix} B_0(1,1;\omega_1) & B_0(1,1;\omega_2) & B_0(1,1;\omega_3) \\ B_1(1,2;\omega_1) & B_1(1,2;\omega_2) & B_1(1,2;\omega_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0.88 & 0.90 & 0.92 \end{bmatrix},$$

where $B_i(t, t+h; \omega_j)$ denotes the price at date t of the ZCB i, maturing at $t+h$, under the j^{th} state of the world ($j \in \{1, 2, 3\}$).

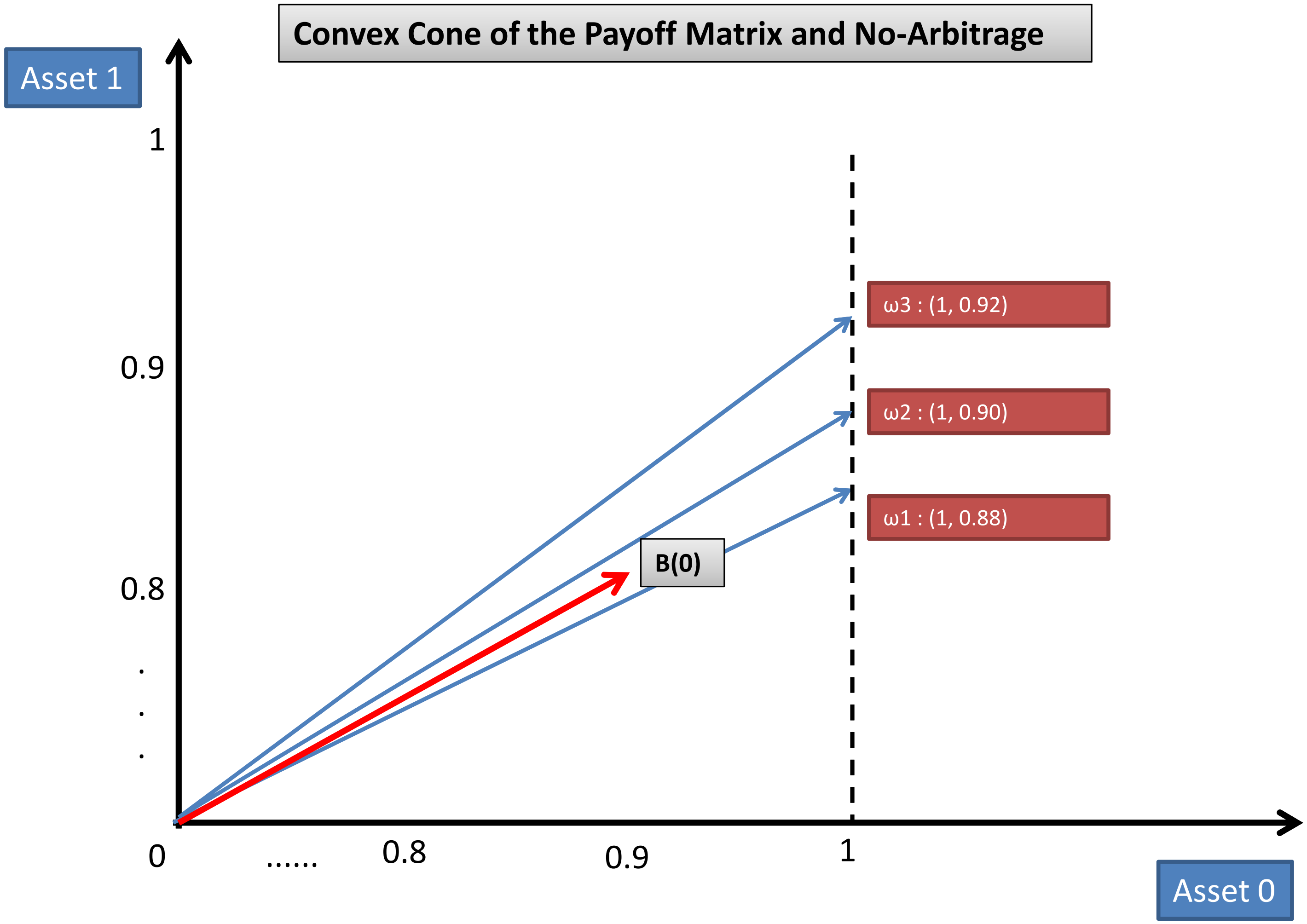
Q.1) Is this market arbitrage-free?

Q.2) Is this market complete ?

Q.3) Are there in the market redundant assets?

$Q.1)$ The vector $B(0) = [0.9, 0.81]'$ of ZCB prices belongs to the convex cone generated by the columns of $\mathbf{B}$ [see next picture].

- This means that there exists at least one positive vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})'$ such that $B(0) = \mathbf{B}\, q^{ad}$,
- and therefore, by the first fundamental theorem of asset pricing the market is arbitrage-free.



Convex Cone of the Payoff Matrix and No-Arbitrage
Asset 1
1
0.9
0.8
ω3 : (1, 0.92)
ω2 : (1, 0.90)
ω1 : (1, 0.88)
B(0)
0
......
0.8
0.9
1
Asset 0

$Q.2)$ $k = rank(\mathbf{B}') = 2 < 3 = N$ and therefore the market is incomplete.

- Equivalently, $\mathbf{B}$ does not admits left inverse given that the number of the states of the world is larger than the number of assets in the market.

$Q.3)$ the payoff matrix $\mathbf{B}$ has only right inverse: indeed, we have 3 states of the world and 2 assets with (clearly!) independent payoffs.

This means that there are no redundant securities.

3.6.6 Risk-free payoffs and risk-neutral probabilities

□ Let us assume that in our financial market $\{S(0), \mathbf{S}\}$ the asset 0, with price $S_0(0)$, is a risk-less bond (with maturity in $t = T$) paying one unit of money in any state $\omega \in \Omega$. This means that $S_0(T, \omega) = 1$ for all $\omega \in \Omega$.

□ Its price is therefore given by $S_0(0) = S_0(T)' q^{(ad)} = \sum_{j=1}^{N} q_j^{(ad)} =: q_0^{(ad)}$, and the associated (continuously compounded) interest rate $r = \frac{1}{T} \ln(1/S_0(0))$ is such that:

$$e^{rT} = \frac{1}{\sum_{j=1}^{N} q_j^{(ad)}} = \frac{1}{q_0^{(ad)}} .$$

☐ Now, given the state price vector $q^{(ad)}$, let us define for any state $\omega_j \in \Omega$ the quantity $q_j = \frac{q_j^{(ad)}}{q_0^{(ad)}}$. We have that $q_j \in (0,1)$ and $\sum_{j=1}^{N} q_j = 1$ and therefore $(q_1, \ldots, q_N)'$ can be seen as probabilities.

☐ Let us thus define a new probability measure on Ω by $\mathbb{Q}(\{\omega_j\}) = q_j > 0$, $j \in \{1, \ldots, N\}$. We can represent the price of any asset i as:

$$S_i(0) = \sum_{j=1}^{N} \frac{q_j^{(ad)}}{q_0^{(ad)}} q_0^{(ad)} S_i(T, \omega_j) = \sum_{j=1}^{N} e^{-rT} q_j \, S_i(T, \omega_j) = E_0^{\mathbb{Q}} \left[e^{-rT} S_i(T) \right]$$

or, equivalently $e^{-r0} S_i(0) = E_0^{\mathbb{Q}} \left[e^{-rT} S_i(T) \right]$,

that is, the **discounted price** processes $e^{-rt} S_i(t)$, $t = 0, T$, are $\mathbb{Q}$-martingales.

☐ Given that $\mathbb{Q}(\{\omega_j\}) > 0$ for all j, as well as $\mathbb{P}(\{\omega_j\})$, then $\mathbb{Q} \sim \mathbb{P}$ (is equivalent to) and for these reasons we call $\mathbb{Q}$ an **equivalent martingale measure**.

☐ Moreover, given that the asset price is the expected (under $\mathbb{Q}$) payoff discounted by the risk-free rate r, $\mathbb{Q}$ is also called **risk-neutral probability measure**.

☐ We can also write the pricing formula as:

$$\frac{S_i(0)}{e^{r0}} = E_0^{\mathbb{Q}}\left[\frac{S_i(T)}{e^{rT}}\right],$$

and we denote $N_t = e^{rt}$ (the one-period money-market account) we find that the normalized price $S_i(t)/N_t$ is a $\mathbb{Q}$-martingale and N_t is the **numeraire** we have chosen in that case.

☐ **More generally :** a numeraire is a non-dividend-paying price process $N = (N_t\,, t \geq 0)$ with $N_0 = 1$. Under the EMM $\mathbb{Q}$, we have that $S_i(t)/N_t$ is a $\mathbb{Q}$-martingale.

☐ Remember that arbitrage opportunities do not depend on the chosen numéraire. The choice of N_t is made in order to facilitate the probability-theoretic analysis in complex asset pricing models. It is made in order to derive (more) tractable pricing formulas.

☐ **Pricing formula under $\mathbb{Q}$:** under the AAO, any payoff $y = y(T) \in \mathbb{R}^N$ has a price given by:

$$y(0) = \mathcal{Q}(y(T)) = E_0^{\mathbb{Q}}\left[e^{-rT} y(T)\right] \ ;$$

given that in general $\mathbb{Q}$ is not unique (being $q^{(ad)}$ not unique), we cannot guarantee the theoretical uniqueness of the price (even if any of them is "reasonable").

☐ What happens in practice : the investors of the financial market $\{S(0), \mathbf{S}\}$, on the basis of their preferences and associated trading, implicitly decide the vector of state prices, and thus the EMM $\mathbb{Q}$, to give a price to $y(T) \in \mathbb{R}^N$.

- The system of preferences reflect how much the investors are worried about any source of risk ($\omega \in \Omega$) determining the payoffs of the risky assets $i \in \{1, \ldots, d\}$.

- 1^{st} **FTAP :** In the financial market $\{S(0), \mathbf{S}\}$ there are no arbitrage opportunities if and only if **there exists a (not unique in general) equivalent martingale measure** $\mathbb{Q}$. In other words : $\{S(0), \mathbf{S}\}$ is arbitrage free if and only if there exists a measure $\mathbb{Q} \sim \mathbb{P}$ making discounted asset prices martingales.

- 2^{nd} **FTAP :** Let us assume that the financial market $\{S(0), \mathbf{S}\}$ admits no arbitrage. There **exists a unique equivalent martingale measure** $\mathbb{Q}$ if and only if the market is complete.

Proposition [see T. Björk (2004), Chapter 3] - The following hold :

- The financial market is arbitrage free if and only if there exists an equivalent martingale measure $\mathbb{Q}$.

- The no-arbitrage financial market is complete if and only if the equivalent martingale measure $\mathbb{Q}$ is unique.

- For any payoff $y(T) \in \mathbb{R}^N$, the only prices which are consistent with the AAO principle are of the form:

$$y(0) = E_0^{\mathbb{Q}}\left[e^{-rT}y(T)\right] \tag{1}$$

  where $\mathbb{Q}$ is an EMM for the underlying market.

- If the market is incomplete, then different choices of EMMs $\mathbb{Q}$ in the pricing formula (1) will generically give rise to different prices.

- If $y(T) \in \mathcal{M}(\mathbf{S})$, even in an incomplete market ($\mathcal{M}(\mathbf{S}) \subset \mathbb{R}^N$), the price in (1) will not depend upon the particular choice of $\mathbb{Q}$ (exercise!):

  - if $k = d + 1 < N$, then $y(0) = \mathcal{Q}(y(T)) = q(y(T)) = y(T)'[R^{(S)}\, S(0)]$, where $q^* = R^{(S)}\, S(0)$ can be seen as a unique (but not positive in general) vector of "state prices" such that $q^* \in \mathcal{M}(\mathbf{S})$.

  - if $k < N$ and $k < d + 1$, then $y(0) = \mathcal{Q}(y(T)) = q(y(T)) = y(T)'[R^{(\bar{S})}\, \bar{S}(0)]$, where $\bar{q}^* = R^{(\bar{S})}\, \bar{S}(0)$ can be seen as a unique (but not positive in general) vector of state prices such that $\bar{q}^* \in \mathcal{M}(\mathbf{S})$.

☐ `Example 2:` Let us consider a financial market over a one-period only. This means that we consider only two dates: $t = 0$ and $t = 1$. At the date $t = 0$ two assets ($i \in \{0, 1\}$, say) are available in the market:

- the first one (the asset $i = 0$) is a risk-free bond maturing at $t = 1$ and with a price $S_0(0) = 0.5$;
- the second one (the asset $i = 1$) is a risky asset with price $S_1(0) = 1$.

- At date $t = 1$ we have $N = 3$ possible states of the world : ω_1, ω_2 and ω_3.

- The payoff matrix of the 2 assets, at date $t = 1$, is the following (2×3)-matrix $\mathbf{S}$ (say):

$$\mathbf{S} = \begin{bmatrix} S_0(1,\omega_1) & S_0(1,\omega_2) & S_0(1,\omega_3) \\ S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix},$$

where $S_i(t,\omega_j)$ denotes the price at date t of the asset i under the j^{th} state of the world ($j \in \{1,2,3\}$).

Q.1) Is this market arbitrage-free ? Q.2) Is it complete ?

Q.3) Is there in the market $\{S(0), \mathbf{S}\}$ at least one equivalent martingale measure ? If yes, how many ?

  - Let us exploit the 1^{st} and 2^{nd} FTAP.

$Q.1)$ The first fundamental theorem of asset pricing tell us that the financial market $\{S(0), \mathbf{S}\}$ is arbitrage-free if and only if there exists a vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ of state prices such that $S(0) = \mathbf{S}q^{ad}$.

- In our case, we have to find a positive solution $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})'$ to the:

$$\begin{cases} \frac{1}{2} &= \quad q_1^{ad} + q_2^{ad} + q_3^{ad} \\ 1 &= q_1^{ad} + 2q_2^{ad} + 4q_3^{ad}\,. \end{cases}$$

- Now, taking q_3^{ad} as a parameter (we have two equations and three unknowns) the solution for q_1^{ad} and q_2^{ad} is:

$$\begin{cases} q_1^{ad} &= \quad 2q_3^{ad} \\ q_2^{ad} &= \frac{1}{2} - 3q_3^{ad}\,. \end{cases}$$

and we have $q_1^{ad} > 0$ and $q_2^{ad} > 0$ if and only if $0 < q_3^{ad} < \frac{1}{6}$.

- This means that, for any positive value of $q_3^{ad} \in]\,0, \frac{1}{6}\,[$ we have a vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ solving the system $S(0) = \mathbf{S}q^{ad}$ and therefore the market is arbitrage-free.

$Q.2)$ The second fundamental theorem of asset pricing tell us that a no-arbitrage market is complete if and only if there exists a unique vector $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad})' \in \mathbb{R}^3_{++}$ of state prices such that $S(0) = \mathbf{S}q^{ad}$.

- From question $i)$ we have seen that we have an infinity of positive solutions q^{ad}, each one associated to a different value of $q_3^{ad} \in]\,0, \frac{1}{6}\,[$. This means that the market is incomplete.

$Q.3$) Given that the market is arbitrage-free and incomplete, we have an infinity of equivalent martingale measures.

- For any $q_3^{ad} \in]\,0, \frac{1}{6}\,[$, the probability measure $\mathbb{Q}$ given by (q_1, q_2, q_3), with $q_j = q_j^{ad} / \sum_{j=1}^{3} q_j^{ad}$ for all $j \in \{1, 2, 3\}$, is an equivalent martingale measure for the market.

- Indeed, from the risk-free asset we have that the (continuously compounded) short rate is $r = \ln(1/S_0(0))$ and therefore $e^r = 1/\sum_{j=1}^{3} q_j^{ad}$. Thus, we can represent the price of the risky asset as:

$$S_1(0) = \sum_{j=1}^{3} \frac{q_j^{(ad)}}{q_0^{(ad)}} q_0^{(ad)} S_1(1, \omega_j) = \sum_{j=1}^{3} e^{-r} q_j \, S_1(1, \omega_j) = E^{\mathbb{Q}} \left[e^{-r} S_1(1) \right] ,$$

proving the fact that $\mathbb{Q}$ is an equivalent martingale measure.

3.6.7 Stochastic discount factors

- A **stochastic discount factor (SDF)** (or state-price deflator, or pricing kernel) is a random variable $m(T,\omega)$, identified also by the N-dimensional vector of its possible realizations $m(T) = (m_1, \ldots, m_N)' \in \mathbb{R}^N$, such that the price of a payoff $y(T)$ is $E_0[m(T,\omega)\, y(T,\omega)]$. Let us start from the LOP:

- **Theorem :** In the financial market $\{S(0), \mathbf{S}\}$ the LOP holds if and only if there is a random variable $m^*(T) \in \mathcal{M}(\mathbf{S})$ such that:

$$q(y) = E_0[m^*(T,\omega)\, y(T,\omega)] = \sum_{j=1}^{N} m_j^*\, y_j\, p_j\,, \ \ \forall\, y \in \mathcal{M}(\mathbf{S})\,.$$

This random variable $m^*(T) \in \mathcal{M}(\mathbf{S})$ is unique but not positive in general.

☐ If $\{S(0), \mathbf{S}\}$ is complete, then each realization of the SDF is given by:

$$m_j^* = \frac{q_j^{(ad)}}{p_j}\,, \ \forall j \in \{1, \ldots, N\}\,.$$

☐ Indeed:

$$q(y) = \sum_{j=1}^{N} y_j\, q(e_j) = \sum_{j=1}^{N} y_j\, q_j^{(ad)} = \sum_{j=1}^{N} y_j\, p_j \underbrace{\frac{q_j^{(ad)}}{p_j}}_{m_j^*}\,.$$

☐ $\forall j \in \{1, \ldots, N\}$, given a unique $q_j^{(ad)}$, we have a unique m_j^*.

☐ If $\{S(0), \mathbf{S}\}$ is incomplete without redundant assets, then:

$$m^*(T,\omega) = S(T,\omega)'\{E[S(T,\omega)S(T,\omega)']\}^{-1}\, S(0)\,;$$

or, stacking the components in the vector $m^*(T)$:

$$m^*(T) = \mathbf{S}'\{E[\mathbf{S}\mathbf{S}']\}^{-1}\, S(0)\,;$$

- Indeed: we are searching for $m^*(T) \in \mathcal{M}(\mathbf{S})$ with the asset span being a linear vector space, i.e. its elements are random variables of the form $S(T,\omega)'\varphi$, with φ an arbitrary portfolio.

- This means that we search for $m^*(T,\omega) = S(T,\omega)'\varphi$ pricing basis assets. That is, we have to construct φ so that:

$$\begin{aligned} S(0) &= E_0[m^*(T,\omega)\,S(T,\omega)] = E_0[S(T,\omega)\,S(T,\omega)'\varphi] \\ &= E_0[S(T,\omega)\,S(T,\omega)']\,\varphi\,, \end{aligned}$$

which requires

$$\varphi = [E(S(T,\omega)S(T,\omega)')]^{-1}\,S(0)\,,$$

- and, thus,

$$m^*(T,\omega) = S(T,\omega)'\{E[S(T,\omega)S(T,\omega)']\}^{-1}\,S(0)\,;$$

$\hookrightarrow$ In terms of state prices, remember that, under the LOP (if $y \in \mathcal{M}(\mathbf{S})$):

$$q(y) \;=\; y'q^* = y'R^{(S)}S(0) = y'\mathbf{S}'(\mathbf{S}\,\mathbf{S}')^{-1}\,S(0)\,.$$

□ If $\{S(0), \mathbf{S}\}$ is incomplete with redundant assets, then:

$$\bar{m}^*(T) = \bar{\mathbf{S}}'[E(\bar{\mathbf{S}}\bar{\mathbf{S}}')]^{-1}\,\bar{S}(0)\,,$$

$\hookrightarrow$ given that, under the LOP (if $y \in \mathcal{M}(\mathbf{S})$):

$$q(y) \;=\; y'\overline{q}^* = y'R^{(\bar{S})}S(0)\,.$$

☐ 1^{st} **FTAP :** In the financial market $\{S(0), \mathbf{S}\}$ there are no arbitrage opportunities if and only if **there exists a (not unique in general) positive** SDF $m(T)$ such that:

$$\begin{aligned} y(0) &= \mathcal{Q}(y(T)) = E_0[m(T,\omega)\, y(T,\omega)]\,, \ \forall \ y(T) \in \mathbb{R}^N\,, \\ m_j &= \frac{q_j^{(ad)}}{p_j}\,, \ q_j^{(ad)} \text{ not unique in general}\,. \end{aligned} \tag{2}$$

☐ 2^{nd} **FTAP :** Let us assume that the financial market $\{S(0), \mathbf{S}\}$ admits no arbitrage. There **exists a unique positive** SDF $m(T,\omega)$ if and only if the market is complete.

- See Cochrane (2005).

Proposition - The following hold :

- The financial market is arbitrage free if and only if there exists positive SDF $m(T, \omega)$.
- The no-arbitrage financial market is complete if and only if the positive SDF $m(T, \omega)$ is unique.
- For any payoff $y(T) \in \mathbb{R}^N$, the only prices which are consistent with the AAO principle are of the form:

$$y(0) = \mathcal{Q}(y(T)) = E_0 \left[m(T, \omega) y(T, \omega) \right] , \tag{3}$$

where $m(T, \omega)$ is a positive SDF for the underlying market.

- If the market is incomplete, then different choices of SDFs $m(T,\omega)$ in the pricing formula (3) will generically give rise to different prices.

- If $y(T) \in \mathcal{M}(\mathbf{S})$, even in an incomplete market ($\mathcal{M}(\mathbf{S}) \subset \mathbb{R}^N$), the price in (3) is unique and given by the unique SDF $m^*(T) \in \mathcal{M}(\mathbf{S})$:

  - if $k = d+1 < N$, then $y(0) = \mathcal{Q}(y(T)) = q(y(T)) = E\left[m^*(T,\omega)y(T,\omega)\right]$, where $m^*(T) = \mathbf{S}'[E(\mathbf{S}\mathbf{S}')]^{-1}\,S(0)$ is the unique (but not positive in general) SDF $m^*(T) \in \mathcal{M}(\mathbf{S})$.

  - if $k < N$ and $k < d+1$, then $y(0) = \mathcal{Q}(y(T)) = q(y(T)) = E\left[\bar{m}^*(T,\omega)y(T,\omega)\right]$, where $\bar{m}^*(T) = \bar{\mathbf{S}}'[E(\bar{\mathbf{S}}\bar{\mathbf{S}}')]^{-1}\,\bar{S}(0)$ is the unique (but not positive in general) SDF $\bar{m}^*(T) \in \mathcal{M}(\mathbf{S})$.

3.6.8 Stochastic discount factor and change of probability measure

- ☐ Recalling the notation $p_j = \mathbb{P}(\{\omega_j\}) > 0$ and $q_j = \mathbb{Q}(\{\omega_j\}) > 0$, for all $j \in \{1, \ldots, N\}$, about the **historical** and **risk-neutral** probability measures, we may define a new random variable on Ω.
- ☐ **Definition :** the random variable L on Ω is defined by $L(\omega_j) = \dfrac{q_j}{p_j}$ for all $j \in \{1, \ldots, N\}$. It is the **likelihood ratio** between the probability measures $\mathbb{Q}$ and $\mathbb{P}$.
- ☐ In more general situations, L is known as the *Radon-Nikodym derivative* of $\mathbb{Q}$ with respect to $\mathbb{P}$: $L(\omega_j) = \dfrac{d\mathbb{Q}}{d\mathbb{P}}(\omega_j) = \dfrac{q_j}{p_j}$.

☐ We have that this **change of probability measure**, from $\mathbb{P}$ to $\mathbb{Q}$ can be written by means of the SDF $m(T)$:

$$L(\omega_j) = \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega_j) = \frac{m(T,\omega_j)}{E_0[m(T,\omega)]}\,, \text{ that is } L(\omega_j) = \frac{q_j}{p_j} = \frac{m_j}{\sum_{j=1}^{N} m_j\, p_j}\,.$$

☐ Now, in the case of a risk-free asset we have:

$$S_0(0) = e^{-r\,T} = E_0[m(T,\omega)]$$

☐ and therefore we can write:

$$L(\omega_j)\, e^{-rT} = m(T,\omega_j)\,, \text{ that is } \frac{q_j}{p_j}\, e^{-rT} = m_j\,.$$

☐ Thus, for any payoff $y(T) \in \mathbb{R}^N$, the only prices which are consistent with the AAO principle are of the form:

$$y(0) = E_0\left[m(T,\omega)y(T,\omega)\right] = E_0^{\mathbb{Q}}\left[e^{-rT}y(T,\omega)\right] \quad (4)$$

where $m(T)$ ($\mathbb{Q}$) is a positive SDF (an EMM) for the underlying market.

☐ If we consider the state-j A-D security with payoff $y(T) = e_j$, we have that its no-arbitrage price can written as:

$$q_j^{(ad)} = E_0\left[m(T,\omega)y(T,\omega)\right] = m_j\,p_j\,,\ \ \forall\, j \in \{1,\ldots,N\}\,.$$

☐ Under the AAO principle, for a given EMM and a given state ω_j, the associated value of positive SDF $m(T,\omega_j)$ can be seen as the A-D price divided by $\mathbb{P}(\{\omega_j\}) = p_j$.

☐ In the previous characterization of the pricing formula, *via* the SDF $m(T)$, we have used as numeraire the process $N_t = e^{r\,t}$. Now, given $m(T)$, we can provide a more general formula for the change of probability measure:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega_j) = \frac{N_T\, m(T,\omega_j)}{N_0} > 0\,, \quad E_0\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1\,, \ \text{ being } \ N_0 = E_0[N_T\, m(T,\omega)]\,,$$

and we find again that the price $y(0)$ is such that $y(0)/N_0$ is a $\mathbb{Q}$-martingale.

☐ Indeed

$$y(0) = E_0[m(T,\omega)\, y(T,\omega)] \Longleftrightarrow \frac{N_0}{N_0} y(0) = E_0\left[\frac{N_T}{N_T}\, m(T,\omega)\, y(T,\omega)\right]$$

☐ thus:

$$\frac{y(0)}{N_0} = E_0\left[\frac{N_T\, m(T,\omega)}{N_0}\frac{y(T,\omega)}{N_T}\right] = E_0^{\mathbb{Q}}\left[\frac{y(T,\omega)}{N_T}\right]\,.$$

3.7 Arbitrage theory in a dynamic discrete-time model

3.7.1 The setup of the discrete-time model

- ☐ The frictionless financial market contains $d+1$ traded basic assets, whose prices at time $t = 0$ are denoted by the vector $S(0) \in \mathbb{R}_+^{d+1}$.
- ☐ Economic activities (trading) take place at dates $t \in \{1, \ldots, T\}$, where T is the terminal date for all these economic activities.
- ☐ Randomness is formalized by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$, that is an non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$. $\mathcal{F}_t$ represents the (common) information available to any investor at date t.

- The prices of the assets at date t are non-negative $\mathcal{F}_t$-measurable random variables that we organize in the vector $S(t) = [S_0(t), \ldots, S_d(t)]'$.
- "$\mathcal{F}_t$-measurable" means that at date t we know (we observe) the prices of the basic assets.
- A **financial asset** (basic asset, derivatives or **contingent claims**) has a price (payoff) $y(t)$ at date t which is a $\mathcal{F}_t$-measurable random variable.
  - If we think about stocks and bonds we have $y(t) = S_i(t)$ and $y(t) = B(t, h)$.
  - In the case of a European Call option we have : $y(T) = (S_i(T) - K)^+$, where $S_i(T)$ is the underlying risky asset (stock, index, ...), T is the maturity date of the contract and K is the strike price. This payoff is $\mathcal{F}_T$-measurable

- Let us denote by $L_{2,t}$ the (Hilbert) space of $\mathcal{F}_t$-measurable square integrable random variables $x(t)$ (say) defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ $(Var[x(t)] < \infty)$.
- The payoff of any financial asset in the market is such that $y(t) \in L_{2,t}$.
- Any payoff $y(t)$ delivered at date t has a price at date $s < t$ for each $\mathcal{F}_t$, denoted by $\mathcal{Q}_s(y(t))$, and function of $\mathcal{F}_s$. In other words, $\mathcal{Q}_s(y(t))$ is an $\mathcal{F}_s$-measurable random variable.
- We also assume the *linearity* and *continuity* of the pricing function $\mathcal{Q}_s(.)$:
  - $\mathcal{Q}_s\left[\lambda_1 y_1(t) + \lambda_2 y_2(t)\right] = \lambda_1 \mathcal{Q}_s(y_1(t)) + \lambda_2 \mathcal{Q}_s(y_2(t))$ (law of one price);
  - if $y_n(t) \xrightarrow[n \to \infty]{L_{2t}} 0$, $\mathcal{Q}_s(y_n(t)) \underset{n\to\infty}{\longrightarrow} 0$, where $\{y_n(t) : n = 1, 2, \ldots\}$ is a date-t sequence of payoffs (converging towards 0).

3.7.2 Self-financing trading strategies

- A **trading strategy** (or **dynamic portfolio**) φ is a $\mathbb{R}^{d+1}$ vector stochastic process $\varphi = (\varphi(t))_{t=1}^{T} = ((\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))')_{t=1}^{T}$ which is **predictable**.
- This means that each $\varphi_i(t)$ is $\mathcal{F}_{t-1}$-measurable for $t \geq 1$.
- Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t BUT to be determined "just before t", i.e. with only knowledge of the information $\mathcal{F}_{t-1}$.
- In particular, the investor decides $\varphi_i(t)$ observing $S_i(t-1)$ and he does not yet know $S_i(t)$.

□ The **value of the portfolio** at time t is the scalar product:

$$S_\varphi(t) = \varphi(t)' S(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t)\,, \;\; t \in \{1, \ldots, T\}$$

and $S_\varphi(0) = \varphi(1)' S(0)\,.$

□ The process $S_\varphi(t)$ is called the **wealth** or the **value process** of the trading strategy φ.

□ The initial wealth $S_\varphi(0)$ is called **initial investment** or **endowment** of the investor.

- Now, given that $\varphi(t)'S(t-1)$ reflects the market value of the portfolio just after it has been established at time $t-1$, and $\varphi(t)'S(t)$ its value just after time t prices are observed (but before changes are made in the portfolio), we have that:

$$\varphi(t)'(S(t) - S(t-1)) = \varphi(t)'\Delta S(t)\,,$$

  is the **change of the market value** due to only security price variations between $t-1$ and t. Thus we can define:

- The **gains process** G_φ of a trading strategy φ is given by:

$$G_\varphi(t) = \sum_{\tau=1}^{t} \varphi(\tau)'(S(\tau) - S(\tau-1)) = \sum_{\tau=1}^{t} \varphi(\tau)'\Delta S(\tau)\,,\ \ t \in \{1, \ldots, T\}\,.$$

☐ Now, let us assume that $S_0(t)$ is the money-market account, that is $S_0(0) = 1$ and $S_0(t) = \exp(r_0 + \ldots + r_{t-1})$. Let us take this asset as numeraire and let us consider the vector of **discounted asset prices** $\tilde{S}(t) = \left(1, \frac{S_1(t)}{S_0(t)}, \ldots, \frac{S_d(t)}{S_0(t)}\right)'$. Then:

☐ The **discounted value process** is:

$$\tilde{S}_\varphi(t) = \frac{1}{S_0(t)}(\varphi(t)'S(t)) = \varphi(t)'\tilde{S}(t)\,, \;\; t \in \{1, \ldots, T\}\,.$$

☐ and the **discounted gains process** is:

$$\tilde{G}_\varphi(t) = \sum_{\tau=1}^{t} \varphi(\tau)'(\tilde{S}(\tau) - \tilde{S}(\tau - 1)) = \sum_{\tau=1}^{t} \varphi(\tau)'\Delta\tilde{S}(\tau)\,, \;\; t \in \{1, \ldots, T\}\,.$$

- The strategy φ is **self-financing**, $\varphi \in \Phi$, if:

$$\varphi(t)'S(t) = \varphi(t+1)'S(t)\,,\ \ t \in \{1, \ldots, T-1\}\,.$$

Interpretation : when new prices $S(t)$ are quoted (observed) at time t, the investor adjust his portfolio from $\varphi(t)$ to $\varphi(t+1)$ without bringing in or consuming any wealth.

- A trading strategy φ is self-financing with respect to $S(t)$ if and only if φ is self-financing with respect to $\tilde{S}(t)$ (exercise).

- A trading strategy φ belongs to Φ if and only if (exercise):

$$\tilde{S}_{\varphi}(t) = \tilde{S}_{\varphi}(0) + \tilde{G}_{\varphi}(t)\,,\ \ t \in \{1, \ldots, T\}\,.$$

3.7.3 The no-arbitrage condition

- Let $\widetilde{\Phi} \subset \Phi$ be a set of self-financing trading strategies. A strategy φ is called an **arbitrage opportunity** or **arbitrage strategy** with respect to $\widetilde{\Phi}$ if:

$$\mathbb{P}\{S_\varphi(0) = 0\} = 1\,,$$

$$\mathbb{P}\{S_\varphi(T) \geq 0\} = 1\,, \text{ and } \mathbb{P}\{S_\varphi(T) > 0\} > 0\,.$$

- An arbitrage opportunity is a self-financing trading strategy with zero initial value, which produces a non-negative final value with probability one and has positive probability of a positive final value.

- We say that the financial market $\{S(0), (S(t))_{t=1}^T\}$ is arbitrage free if there are no arbitrage opportunities in the class Φ of trading strategies.

☐ We say that the **absence of arbitrage opportunity principle** in satisfied in the financial market if at any $t \in \{0, \ldots, T-1\}$ it is impossible to constitute a portfolio $\varphi \in \Phi$, possibly modified at subsequent dates, such that:

i) its price at t is non positive;

ii) its payoffs at subsequent dates are non negative;

iii) there exists at least one date $s > t$ such that the net payoff, at s, is strictly positive with a strictly positive conditional probability at t.

- ☐ A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ is called a martingale measure for $\tilde{S} = (\tilde{S}(t))_{t=1}^T$ if the process $\tilde{S} = (\tilde{S}(t))_{t=1}^T$ follows a $\mathbb{Q}$-martingale with respect to the filtration $\mathbb{F}$.

- ☐ Let $\mathbb{Q}$ an EMM and $\varphi \in \Phi$ any self-financing strategy. The the wealth process $\tilde{S}_\varphi(t)$ is a $\mathbb{Q}$-martingale with respect to the filtration $\mathbb{F}$ (exercise).

- ☐ 1^{st} **FTAP :** The financial market $\{S(0), (S(t))_{t=1}^T\}$ is arbitrage-free if and only if **there exists a (not unique in general) equivalent martingale measure** $\mathbb{Q}$. Equivalently : $\{S(0), (S(t))_{t=1}^T\}$ is arbitrage-free if and only if there exists a measure $\mathbb{Q} \sim \mathbb{P}$ making the $d-$dimensional discounted asset price process $\tilde{S} = (\tilde{S}(t))_{t=1}^T$ a martingale.

3.7.4 Attainable payoffs

□ Given the market $\{S(0), (S(t))_{t=1}^{T}\}$, a financial asset with payoff $y(T)$ is **attainable** (i.e. it is in the *asset span*) is there exists a **replicating strategy** $\varphi \in \Phi$ such that:

$$S_\varphi(T) = y(T)\,, \text{ or, equivalently}$$

$$\frac{S_\varphi(T)}{S_0(T)} = \tilde{S}_\varphi(T) = S_\varphi(0) + \tilde{G}_\varphi(T) = \frac{y(T)}{S_0(T)}\,.$$

□ If $y(T)$ is attainable, then the no-arbitrage price process $y = (y(t))_{t=0}^{T}$ is given by the value process of *any* replicating strategy φ for $y(T)$:

$$\begin{aligned}
\mathcal{Q}_t(y(T)) &= S_\varphi(t) = S_0(t)\tilde{S}_\varphi(t) \\
&= S_0(t)\, E^{\mathbb{Q}}\left[\tilde{S}_\varphi(T) \,|\, \mathcal{F}_t\right] , \text{ as } \tilde{S}_\varphi(t) \text{ is a } \mathbb{Q}\text{-martingale} \\
&= S_0(t)\, E^{\mathbb{Q}}\left[S_0^{-1}(T)\, S_\varphi(T) \,|\, \mathcal{F}_t\right] \\
&= S_0(t)\, E^{\mathbb{Q}}\left[S_0^{-1}(T)\, y(T) \,|\, \mathcal{F}_t\right] , \text{ as } \varphi \text{ is a replicating strategy for } y(T) \\
&= E^{\mathbb{Q}}\left[e^{-r_t-\cdots-r_{T-1}}\, y(T) \,|\, \mathcal{F}_t\right] \\
&= E_t^{\mathbb{Q}}\left[e^{-r_t-\cdots-r_{T-1}}\, y(T)\right] .
\end{aligned}$$

3.7.5 Complete markets: uniqueness of the EMM

- ☐ The financial market $\{S(0), (S(t))_{t=1}^{T}\}$ is **complete** if every contingent claim with payoff $y(T)$ is attainable, i.e. for every $\mathcal{F}_T$-measurable random variable $y(T)$ there exists a replicating self-financing trading strategy $\varphi \in \Phi$ such that $S_\varphi(T) = y(T)$.
- ☐ 2^{nd} **FTAP :** Let us assume that the financial market $\{S(0), (S(t))_{t=1}^{T}\}$ admits no arbitrage. There **exists a unique equivalent martingale measure** $\mathbb{Q}$ if and only if the market is complete. Equivalently: the arbitrage-free market $\{S(0), (S(t))_{t=1}^{T}\}$ is complete if and only if there exists a measure $\mathbb{Q} \sim \mathbb{P}$ making the $d-$dimensional discounted asset price process $\tilde{S} = (\tilde{S}(t))_{t=1}^{T}$ a martingale.

Proposition - The following hold :

- The financial market is arbitrage free if and only if there exists an equivalent martingale measure $\mathbb{Q}$.
- The no-arbitrage financial market is complete if and only if the equivalent martingale measure $\mathbb{Q}$ is unique.
- For any payoff (random variable) $y(T)$, the only prices which are consistent with the AAO principle are of the form:

$$y(t) = \mathcal{Q}_t(y(T)) = E_t^{\mathbb{Q}}\left[e^{-r_t - \ldots - r_{T-1}} y(T)\right] \tag{5}$$

where $\mathbb{Q}$ is an EMM for the underlying market.

- If the market is incomplete, then different choices of EMMs $\mathbb{Q}$ in the pricing formula (5) will generically give rise to different no-arbitrage prices.
- If $y(T)$ is attainable, then the no-arbitrage price process $y = (y(t))_{t=0}^{T}$ is given by the value process of any replicating strategy φ for $y(T)$.

3.7.6 Stochastic discount factors

- The **stochastic discount factor (SDF)** (or state-price deflator, or pricing kernel) $m(t, T)$ is a $\mathcal{F}_T$-measurable random variable such that the price at date t of any $\mathcal{F}_T$-measurable payoff $y(T)$ can be represented as $E_t[m(t, T)\, y(T)]$.
- If the financial market $\{S(0), (S(t))_{t=1}^{T}\}$ is **arbitrage-free**, then there exists a positive $\mathcal{F}_{t+1}$-measurable random variable $m(t, t+1)$ such that the price $y(t)$ at date t of any asset that does not pay any dividend at $t+1$ satisfies:

$$y(t) = E_t[m(t, t+1) y(t+1)] \,.$$

☐ More generally for any payoff $y(t+h)$ at $t+h \leq T$, we have :

$$y(t) \quad = \quad E_t[m(t,t+h)y(t+h)] = E_t[m(t,t+1)\dots m(t+h-1,t+h)y(t+h)]\,.$$

☐ Thus, we have that:

$$m(t,T) = m(t,t+1)m(t+1,t+2)\cdots m(T-1,T) = \frac{m(0,T)}{m(0,t)}.$$

We will call $m(t,t+1)$ ($m(t,t+h)$, respectively) the one-period (h-period, respectively) stochastic discount factor, while $m(0,T)$ will be named *state price deflator*.

☐ We can also write no-arbitrage price process $y = (y(t))_{t=0}^{T}$ as:

$$m(0,t)y(t) = E_t[m(0,T)y(T)]\,, \text{ so } m(0,t)y(t) \text{ is a } \mathbb{P}\text{-martingale.}$$

- 1^{st} **FTAP :** The financial market $\{S(0), (S(t))_{t=1}^{T}\}$ is arbitrage-free if and only if **there exists a (not unique in general) positive** SDF $m(t,T)$ such that the price $y(t)$ at date t of any payoff $y(T)$ can be written as:

$$y(t) = E_t[m(t,T)y(T)]\,.$$

- 2^{nd} **FTAP :** Let us assume that the financial market $\{S(0), (S(t))_{t=1}^{T}\}$ admits no arbitrage. There **exists a unique positive** SDF $m(t,T)$ such that the price at date t of any $\mathcal{F}_T$-measurable payoff $y(T)$ is given by $E_t[m(t,T)\,y(T)]$ if and only if the market is complete.

Proposition - The following hold :

- The financial market is arbitrage free if and only if there exists positive SDF $m(0,T)$.
- The no-arbitrage financial market is complete if and only if the positive SDF $m(0,T)$ is unique.
- For any payoff $y(T)$, the only date t prices which are consistent with the AAO principle are of the form:

$$y(t) = \mathcal{Q}_t(y(T)) = E_t\left[m(t,T)y(T)\right], \ \forall t \in 0,\ldots,T-1 \tag{6}$$

where $m(t,T)$ is a positive SDF for the underlying market.

- If the market is incomplete, then different choices of SDFs $m(t,T)$ in the pricing formula (6) will generically give rise to different prices.

- If $y(T)$ is attainable, even in an incomplete market, the price in (6) is unique and given by the unique SDF $m^*(t,T)$ in the asset span.

3.7.7 Stochastic discount factor and change of probability measure

- ☐ A numeraire is defined as a non-dividend-paying price process $N = (N_t,\, t \geq 0)$ with $N_0 = 1$. In other words, N is a stochastic process such that, for every $T > t$:

$$N_t = E_t[m(t,T)\, N_T]\,, \text{ and } N_0 = E_0[m(0,T)\, N_T] = 1\,, \text{ where}$$

$$m(t,T) = m(t,t+1) \cdot \ldots \cdot m(T-1,T)\,.$$

- ☐ The process $N^* = (N_t\, m(0,t),\, t \geq 0)$ is therefore a $\mathbb{P}$-martingale with unitary value in $t = 0$. Let $\mathbb{Q}$ be the probability (equivalent to $\mathbb{P}$) defined by the sequence of conditional densities:

$$\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{N_{t+1}\, m(t,t+1)}{N_t} > 0\,, \;\; E_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}}\right] = 1\,, t \in \{0, \ldots, T-1\}\,.$$

☐ This means that $\dfrac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}}$ is the one-period conditional p.d.f. of $\mathbb{Q}$ w.r.t. to $\mathbb{P}$, and the associated Radon-Nikodym derivative is:

$$\xi_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=0}^{T-1} \frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{N_T\, m(0,T)}{N_0} = \prod_{t=0}^{T-1} \frac{N_{t+1}\, m(t,t+1)}{N_t}.$$

This means that ξ_T is the joint p.d.f. (over the time period of interest $\{0,\ldots,T\}$) of $\mathbb{Q}$ w.r.t. to $\mathbb{P}$.

☐ Observe that we have:

$$\xi_t = \prod_{i=0}^{t-1} \frac{d\mathbb{Q}_{i,i+1}}{d\mathbb{P}_{i,i+1}} = E_t(\xi_T),$$

and therefore the ξ_t is a $\mathbb{P}$-martingale.

- We have that a price process $y(t)$ is such that $y(t)/N_t$ is a $\mathbb{Q}$-martingale:

$$y(t) = E_t[m(t,t+1)y(t+1)] \Longleftrightarrow \frac{N_t}{N_t}y(t) = E_t\left[\frac{N_{t+1}}{N_{t+1}}m(t,t+1)\,y(t+1)\right]$$

- thus:

$$\frac{y(t)}{N_t} = E_t\left[\frac{N_{t+1}m(t,t+1)}{N_t}\frac{y(t+1)}{N_{t+1}}\right] = E_t^{\mathbb{Q}}\left[\frac{y(t+1)}{N_{t+1}}\right].$$

- Observe that the *no-arbitrage price* at date t of a ZCB with maturity date in $t+1$ is such that:

$$B(t,t+1) = E_t\left[m(t,t+1)\right] = e^{-r_t}$$

where r_t is the $(t,t+1)$ short rate (YTM), known in t.

□ If we consider as numeraire the money-market account $N_t = \exp(r_0 + \ldots + r_{t-1}) = A_{0,t}$, where $(A_{0,t})^{-1} = E_0(m(0,1)) \cdots E_{t-1}(m(t-1,t))$, the associated equivalent probability $\mathbb{Q}$ has a one-period conditional density, with respect to $\mathbb{P}$, given by :

$$\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{A_{0,t+1} m(t,t+1)}{A_{0,t}} = \frac{m(t,t+1)}{E_t(m(t,t+1))} = e^{r_t} m(t,t+1)\,.$$

and it is called **risk-neutral probability measure**.

□ This means that the pricing formula $y(t) = E_t[m(t,t+1)y(t+1)]$ can be written:

$$\begin{aligned} y(t) &= E_t\left[\frac{m(t,t+1)}{E_t[m(t,t+1)]} E_t[m(t,t+1)] y(t+1)\right] \\ &= E_t^{\mathbb{Q}}[\exp(-r_t) y(t+1)]\,. \end{aligned}$$

- So, for any no-arbitrage price process $y = (y(t))_{t=0}^{T}$ we have (denoting $\ln[y(t+1)/y(t)] = r_{t,t+1}$):

$$1 = E_t^{\mathbb{Q}}\left[\exp(-r_t)\frac{y(t+1)}{y(t)}\right] \quad \Rightarrow \exp(r_t) = E_t^{\mathbb{Q}}\left[\exp(r_{t,t+1})\right] .$$

- In a general $(T-t)$-period horizon, the conditional (to $\mathcal{F}_t$) density of the risk-neutral probability $\mathbb{Q}$ with respect to the historical probability $\mathbb{P}$ is given by:

$$\begin{aligned} \frac{d\mathbb{Q}_{t,T}}{d\mathbb{P}_{t,T}} &= \frac{m(t,t+1)\cdot\ldots\cdot m(T-1,T)}{E_t(m(t,t+1))\cdot\ldots\cdot E_{T-1}(m(T-1,T))} \\ &= \exp(r_t+\ldots+r_{T-1})\, m(t,T)\,, \end{aligned}$$

- This means that, for any payoff $y(T)$ at T, we have :

$$y(t) = E_t^{\mathbb{Q}}[\exp(-r_t-\ldots-r_{T-1})y(T)]\,,$$

and $y(t)/A_{0,t}$ is a $\mathbb{Q}$-martingale.

□ In the case of a ZCB maturing at $T = t + h$, its no-arbitrage price at date t is given, under $\mathbb{P}$, by :

$$\begin{aligned} B(t, t+h) &= E_t[m(t, t+h)] \\ &= E_t[m(t, t+1)B(t+1, t+h)]\,. \end{aligned}$$

□ Under the risk-neutral probability $\mathbb{Q}$ we can equivalently write:

$$\begin{aligned} B(t, t+h) &= E_t^{\mathbb{Q}}[\exp(-r_t - \ldots - r_{t+h-1})] \\ &= E_t^{\mathbb{Q}}[\exp(-r_t)B(t+1, t+h)]\,. \end{aligned}$$