Fixed Income and Credit Risk : exercise sheet n° 03

Fall Semester 2012

Professor	Assistant	Program
Fulvio Pegoraro	Roberto Marfè	MSc. Finance

Exercise N° 01 [Replicating strategies in the one-period model].

Let us consider the one-period financial market $\{S(0), \mathbf{S}\}$ where $S(0) \in \mathbb{R}_+^{d+1}$, $S(0) = [S_0(0), S_1(0), \ldots, S_d(0)]'$, is the date $t = 0$ vector of basic asset prices. $\mathbf{S}$ denotes the $[(d+1) \times N]$-matrix of payoffs. Show that:

a) when $k = rank(\mathbf{S}') = N = d+1$ (complete markets without redundant assets), for any payoff $y \in \mathbb{R}^N$ there always exists a unique replicating strategy $\varphi \in \mathbb{R}^{d+1}$;

b) when $k = rank(\mathbf{S}') = N < d+1$ (complete markets with redundant assets), for any payoff $y \in \mathbb{R}^N$ there always exists a replicating strategy but it is not unique;

c) when $k = rank(\mathbf{S}') = d+1 < N$ (incomplete markets without redundant assets), for a given payoff $y \in \mathbb{R}^N$ if there exists a replicating strategy it is unique.

d) when $k = rank(\mathbf{S}') < d+1$ and $k < N$ (incomplete markets with redundant assets), for a given payoff $y \in \mathbb{R}^N$ if there exists a replicating strategy it is not unique.

In each of the four cases indicate under which conditions a replicating strategy exists and, in that case, provide the general formula for the replicating portfolio $\varphi \in \mathbb{R}^{d+1}$.

Exercise N° 02 [Complete financial markets].

Prove the following theorem :

The financial market $\{S(0), \mathbf{S}\}$ is complete if and only if $\mathbf{S}$ admits left inverse, namely $L^{(S)} = (\mathbf{S}'\mathbf{S})^{-1}\,\mathbf{S}'$.

Exercise N° 03 [The law of one price].

The law of one price (LOP) states that if $\mathbf{S}'\varphi^* = \mathbf{S}'\varphi^{**}$, with $\varphi^* \neq \varphi^{**}$ and $\varphi^*, \varphi^{**} \in \mathbb{R}^{d+1}$, then $S_{\varphi^*}(0) = S_{\varphi^{**}}(0)$. Show that the LOP holds if and only if the payoff pricing function $q(.)$ is a linear (single-valued) function of the asset span $\mathcal{M}(\mathbf{S})$.

Exercise N° 04 [Pricing payoffs in the asset span].

Let us assume that the LOP holds. Show that, in an incomplete market without redundant assets ($k = rank(\mathbf{S}') = d+1 < N$) any payoff $y \in \mathcal{M}(\mathbf{S})$ can be priced by the following formula: $q(y) = y'\,R^{(S)}\,S(0) = S(0)'\,L^{(S')}\,y$. Then, let us consider an incomplete market with redundant securities ($k < d+1$, $k < N$). Let us denote with $\bar{\mathbf{S}}$ the (k, N) payoff matrix of the no redundant assets and with $\bar{S}(0)$ the vector of these k asset prices. Show that any payoff $y \in \mathcal{M}(\mathbf{S})$ can be priced by the following formula: $q(y) = y' \, R^{(\bar{S})} \, \bar{S}(0) = \bar{S}(0)' \, L^{(\bar{S}')} \, y$.

Exercise N° 05 [First Fundamental Theorem of Asset Pricing].

Prove the following theorem : in the financial market $\{S(0), \mathbf{S}\}$ there are no arbitrage opportunities if and only if there exists a strictly positive vector of state prices $q^{(ad)} \in \mathbb{R}^N_{++}$ such that:

$$S(0) = \mathbf{S} \, q^{(ad)} .$$

Exercise N° 06.

Let us consider a financial market, with $N = 5$ states of natures, over a one-period only. This means that we consider only two dates: $t = 0$ and $t = 1$. Let us imagine to observe at $t = 0$ a unique vector of Arrow-Debreu security prices (the market is complete): $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad}, q_4^{ad}, q_5^{ad})' = (0.1225, 0.2451, 0.3676, 0.0613, 0.1838)'$.

(i) Compute the price of a risk-free bond paying one unit of money regardless the state of the nature, and its continuously compounded interest rate r.

(ii) Calculate the risk-neutral probabilities associated to q^{ad}.

(iii) Calculate the price of an asset α (say) with payoff at date $t = 1$ given by $S_\alpha(t = 1) = (5, 5, 2, 7, 4)'$.

Exercise N° 07.

Let us consider a financial market over a one-period only. This means that we consider only two dates: $t = 0$ and $t = 1$. At the date $t = 0$ four assets ($i \in \{0, 1, 2, 3\}$, say) are available in the market: the first one (the asset $i = 0$) is a risk-free bond maturing at $t = 1$ and with a price $S_0(0) = 1$. The remaining three assets (assets $i = 1, 2, 3$) are risky assets with prices respectively given by $S_1(0) = 2$, $S_2(0) = 0.6$ and $S_3(0) = 1$. At date $t = 1$ we have $N = 3$ possible states of the world : ω_1, ω_2 and ω_3. The payoff matrix of the 4 assets, at date $t = 1$, is the following (4×3)-matrix $\mathbf{S}$ (say):

$$\mathbf{S} = \begin{bmatrix} S_0(1,\omega_1) & S_0(1,\omega_2) & S_0(1,\omega_3) \\ S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \\ S_2(1,\omega_1) & S_2(1,\omega_2) & S_2(1,\omega_3) \\ S_3(1,\omega_1) & S_3(1,\omega_2) & S_3(1,\omega_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1.5 & 0.5 & 0 \\ 2 & 1 & 0 \end{bmatrix} ,$$

where $S_i(t, \omega_j)$ denotes the price at date t of the asset i under the j^{th} state of the world ($j \in \{1, 2, 3\}$).

$i)$ Is the market complete ? Motivate your answer.

$ii)$ Do we find in this market redundant assets ? How many ? Motivate your answer.

$iii)$ Is this market arbitrage free ? Motivate your answer on the basis of the existence of a vector of strictly positive state prices.

Exercise N° 08.

Let us consider again a financial market over a one-period only. This means that we consider only two dates: $t = 0$ and $t = 1$. At the date $t = 0$ three basis assets ($i \in \{0, 1, 2\}$, say) are available in the market: the first one (the asset $i = 0$) is a risk-free bond maturing at $t = 1$ and with a price $S_0(0) = 2$. The remaining two assets (assets $i = 1, 2$) are risky assets with prices respectively given by $S_1(0) = 1/3$ and $S_2(0) = 1/2$. At date $t = 1$ we have $N = 4$ possible states of the world : ω_1, ω_2, ω_3 and ω_4. The payoff matrix of the 3 assets, at date $t = 1$, is the following (3×4)-matrix $\mathbf{S}$ (say):

$$\mathbf{S} = \begin{bmatrix} S_0(1,\omega_1) & S_0(1,\omega_2) & S_0(1,\omega_3) & S_0(1,\omega_4) \\ S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) & S_1(1,\omega_4) \\ S_2(1,\omega_1) & S_2(1,\omega_2) & S_2(1,\omega_3) & S_2(1,\omega_4) \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix},$$

where $S_i(t, \omega_j)$ denotes the price at date t of the asset i under the j^{th} state of the world ($j \in \{1, 2, 3, 4\}$).

i) What is the (continuously compounded) short rate of the risk-free asset ?

ii) Find all vectors of state prices $q^{ad} = (q_1^{ad}, q_2^{ad}, q_3^{ad}, q_4^{ad})'$ such that $S(0) = \mathbf{S}\, q^{ad}$, where $S(0) = (S_0(0), S_1(0), S_2(0))'$ [consider q_4^{ad} as arbitrary]. Is this market arbitrage free ? Motivate your answer.

iii) Is the market complete ? Motivate your answer.

iv) What are the risk-neutral probabilities ?

v) Let us introduce in the market an additional (fourth) asset with payoff

$$(S_3(1,\omega_1), S_3(1,\omega_2), S_3(1,\omega_3), S_3(1,\omega_4))' = (1, 2, 0, 1)',$$

without affecting the prices and payoffs of the other assets. What are the possible no-arbitrage prices of this new asset ? Consequently, what can we say about this new asset ?

Exercise N° 09 [Numeraire invariance of the self-financing trading strategy].

Prove the following sentence: a trading strategy φ is self-financing with respect to $S(t)$ if and only if φ is self-financing with respect to $\tilde{S}(t) = S(t)/S_0(t)$ where $S_0(t) = \exp(r_0 + \ldots + r_{t-1})$ is the money-market account considered as numeraire.

Exercise N° 10 [Discounted value process and self-financing trading strategy].

Prove the following sentence: A trading strategy φ belongs to Φ (i.e., it is self-financing) if and only if:

$$\tilde{S}_\varphi(t) = \tilde{S}_\varphi(0) + \tilde{G}_\varphi(t)\,, \ \ t \in \{1, \ldots, T\}\,.$$

Exercise N° 11 [Discounted value process and equivalent martingale measure].

Prove the following sentence: let $\mathbb{Q}$ an EMM and $\varphi \in \Phi$ any self-financing strategy. Then the discounted value process $\tilde{S}_\varphi(t)$ is a $\mathbb{Q}$-martingale with respect to the filtration $\mathbb{F}$.