Fixed Income and Credit Risk : solutions for exercise sheet n° 02

Fall Semester 2012

Professor	Assistant	Program
Fulvio Pegoraro	Roberto Marfè	MSc. Finance

Exercise N° 01.

The coupon bond price at date $t = 0$ is given by:

$$CB(0,T) = \sum_{i=1}^{T} (C_i) \times (1+Y)^{-i},$$

and the associated duration D is given by:

$$D = \frac{\sum_{i=1}^{T} (i \times C_i) \times (1+Y)^{-i}}{\sum_{i=1}^{T} (C_i) \times (1+Y)^{-i}}.$$

The numerator is given by $\sum_{i=1}^{5} (i \times C_i) \times (1+0.1)^{-i} = 395.6814$. The denominator, that is the bond price, is $\sum_{i=1}^{5} (C_i) \times (1+0.1)^{-i} = 92.4184$. The duration of the bond is therefore $D = 395.6814/92.4184 = 4.28$ years.

Exercise N° 02 (Exercise N° 01, continued).

We simply need to apply the formula $\frac{dCB(0,T)}{CB(0,T)} = -\frac{D}{1+Y} dY$. In our case we have:

$$\frac{dCB(0,T)}{92.4184} = -\frac{4.28}{1+0.1} \times 0.001 \Rightarrow dCB(0,T) = -0.3596;$$

This means that the new price, predicted using the Duration of the bond, is $92.4184 - 0.3596 = 92.0588$. The *true bond price*, after the interest rate variation, is 92.0596. This means that the true and predicted variation are quite close.

Now, if we consider a rise of the interest rate at 15%, the bond price variation calculated using the its Duration is:

$$\frac{dCB(0,T)}{92.4184} = -\frac{4.28}{1+0.1} \times 0.05 \Rightarrow dCB(0,T) = -17.9796,$$

while its true variation is $76.5349 - 92.4184 = -15.8835$. The true variation is quite smaller than the one approximated using the bond Duration. We numerically observe that the Duration provide a good approximation of the bond price variation only for infinitesimal interest rate variations (like a $dY = 0.001$).

Exercise N° 03.

We have seen during Lecture 1 that, if we consider at date $t = 0$ a coupon bond with a constant coupon rate c, a face value of C_T and time to maturity T, then we have:

$$D = \frac{CB^c(0,T)}{CB(0,T)}\left(1+\frac{1}{Y}\right) + \frac{CB^{C_T}(0,T)}{CB(0,T)}\left(1-\frac{c}{Y}\right)T\,,$$

$$\text{with } CB^c(0,T) = \sum_{i=1}^{T}(c \times C_T) \times (1+Y)^{-i} = (c \times C_T)\frac{1-(1+Y)^{-T}}{Y}\,,$$

$$\text{and } CB^{C_T}(0,T) = C_T(1+Y)^{-T}\,.$$

The perpetual bond (also called perpetuity) has $T = +\infty$, no final repayment and constant coupon of C. This means that $CB^{C_T} = 0$ given that $C_T = 0$ and thus the duration of the perpetuity D_p (say) is:

$$D_p = \frac{CB^c(0,T)}{CB(0,T)}\left(1+\frac{1}{Y}\right) = \frac{1+Y}{Y}$$

given that $CB = CB^c$.

Exercise N° 04.

Given the formula:

$$D = \frac{CB^c(0,T)}{CB(0,T)}\left(1+\frac{1}{Y}\right) + \frac{CB^{C_T}(0,T)}{CB(0,T)}\left(1-\frac{c}{Y}\right)T\,,$$

$$\text{with } CB^c(0,T) = \sum_{i=1}^{T}(c \times C_T) \times (1+Y)^{-i} = (c \times C_T)\frac{1-(1+Y)^{-T}}{Y}\,,$$

$$\text{and } CB^{C_T}(0,T) = C_T(1+Y)^{-T}\,,$$

we have that:
i) the duration of the zero-coupon bond is equal to its residual maturity, that is 7 years;

ii) from the formula $D_p = (1+Y)/Y$ we have that $D_p = 1.05/0.05 = 21$ years (the coupon information is irrelevant);

iii) this coupon bond is a par bond ($c = Y$ and thus $CB(0,T) = C_T$) and therefore the Duration formula is:

$$D = (1-(1+Y)^{-T}) \times \left(1+\frac{1}{Y}\right)\,.$$

This means that $D = [(1 - (1.05)^{-10})/(0.05)] \times 1.05 = 8.11$ years.

Exercise N° 05.

The duration of the bond is $D = 416.99/100 = 4.1699$ years. The convexity of the bond is:

$$\begin{aligned}\kappa &= \frac{\displaystyle\sum_{i=1}^{T} \frac{(i \times (1+i) \times C_i)}{(1+Y)^{i+2}}}{\displaystyle\sum_{i=1}^{T} \frac{C_i}{(1+Y)^i}}\,, \\ &= \frac{2343.57}{100 \times (1.1)^2} = 19.3683 \ \ (years)^2\end{aligned}$$

Exercise N° 06 (Exercise N° 05, continued).

We have seen during Lecture 1 that the bond price variation can be expressed in the following way:

$$\frac{\Delta CB(Y)}{CB(Y)} = -D_{mod}\Delta Y + \frac{\kappa}{2}(\Delta Y)^2\,,$$

$$\text{where } D_{mod} = -\frac{dCB(Y)}{dY}\frac{1}{CB(Y)} \ \text{ or } D_{mod} = \frac{D}{1+Y},$$

$$\text{and } \kappa = \frac{1}{CB(Y)}\frac{d^2CB(Y)}{dY^2} \ \text{ is the } \textit{convexity}.$$

In our case we have:

$$\begin{aligned}\frac{\Delta CB(Y)}{CB(Y)} &= -\frac{D}{1+Y}\Delta Y + \frac{\kappa}{2}(\Delta Y)^2 \\ &= -\frac{4.1699}{1.1} \times 0.01 + \frac{19.3683}{2} \times (0.01)^2 = -3.7908\% + 0.0968\% = -3.6940\%\,.\end{aligned}$$

The new price is therefore 96.306.

Exercise N° 07.

The price, duration and convexity can be easily calculated through the following **Table 1** :

Period i	Time T_i	Cash Flow CF	Discount $B(0,T_i)$	Discounted CF $CF \times B(0,T_i)$	Weight w_i	Weight $\times T_i$ $w_i \times T_i$	Weight $\times T_i^2$ $w_i \times T_i^2$
1	0.5	2.5	0.9778	2.44	0.024	0.0118	0.0059
2	1.0	2.5	0.9560	2.39	0.023	0.0231	0.0231
3	1.5	2.5	0.9347	2.34	0.023	0.0338	0.0508
4	2.0	2.5	0.9139	2.28	0.022	0.0441	0.0882
5	2.5	2.5	0.8936	2.23	0.022	0.0539	0.1348
6	3.0	2.5	0.8737	2.18	0.021	0.0633	0.1898
7	3.5	2.5	0.8543	2.14	0.021	0.0722	0.2526
8	4.0	2.5	0.8353	2.09	0.020	0.0806	0.3226
9	4.5	2.5	0.8167	2.04	0.020	0.0887	0.3992
10	5.0	2.5	0.7985	2.00	0.019	0.0964	0.4818
11	5.5	2.5	0.7808	1.95	0.019	0.1036	0.5701
12	6.0	2.5	0.7634	1.91	0.018	0.1106	0.6633
13	6.5	2.5	0.7464	1.87	0.018	0.1171	0.7612
14	7.0	2.5	0.7298	1.82	0.018	0.1233	0.8631
15	7.5	2.5	0.7136	1.78	0.017	0.1292	0.9688
16	8.0	2.5	0.6977	1.74	0.017	0.1347	1.0778
17	8.5	2.5	0.6822	1.71	0.016	0.1400	1.1896
18	9.0	2.5	0.6670	1.67	0.016	0.1449	1.3040
19	9.5	2.5	0.6521	1.63	0.016	0.1495	1.4206
20	10.0	102.5	0.6376	65.36	0.631	6.3101	63.1010
				$CB(0,T) = 103.58$		$D^*_{mod,cb} = 8.03$	$\kappa^*_{cb} = 73.87$

Exercise N° 08 (Exercise N° 07, continued; duration hedging strategy).

We must have $d\Pi = 0$ and, from the relation $\Pi = CB(0,T) + K \times B(0,T)$, we immediately find that this condition implies:

$$K^* = -\frac{D^*_{mod,cb} \times CB(0,T)}{D^*_{mod,zcb} \times B(0,T)} = -\frac{8.03 \times 103.58}{10 \times 63.76} = -1.3045\,.$$

That is, to hedge against parallel shift of the yield curve (interest rate risk), the corporation must short 1.3045 units the 10-year ZCB.

In order to understand how the duration hedge perform in the three scenarios we need to recompute the values of the coupon bond and ZCB for the new interest rate scenarios, compute the new value of the portfolio Π and then take the difference from the original portfolio value. These results are organized in the following **Table 2**:

Yield Curve Shift	$CB(0,T)$	$B(0,T)$	K	$d\Pi$
Initial Values	103.58	63.76	-1.3045	
$dR = 0.1\%$	102.75	63.13	-1.3045	-0.0003
$dR = 1\%$	95.63	57.69	-1.3045	-0.0318
$dR = 2\%$	88.38	52.20	-1.3045	-0.1210
$dR = -0.1\%$	104.41	64.40	-1.3045	-0.0003
$dR = -1\%$	112.29	70.47	-1.3045	-0.0350
$dR = -2\%$	121.84	77.88	-1.3045	-0.1474

As can be seen the change in portfolio value is extremely small in the first scenario, however a larger shift in the yield curve still produces a loss which increases with the interest rate rise. Indeed, the above condition K^* implies a portfolio duration $D_{mod,\Pi} = 0$ clearly highlighting that this strategy is useful for infinitesimal interest rates variations. In addition, observe that the portfolio loses money both when interest rates increase and decrease.

Exercise N° 09 (Exercise N° 08, continued; duration-convexity hedging strategy).

The portfolio is hedged if a shift in the flat yield curve does not affect the portfolio value, that is, if $d\Pi = 0$. Taking into account also the convexity term, we have:

$$\begin{aligned} d\Pi &= dP + K_1 \times dP_1 + K_2 \times dP_2 \\ &= -D^*_{mod} \times P \times dR + \frac{1}{2} \times \kappa^* \times P \times (dR)^2 \\ &\qquad -K_1 \times D^*_{mod,1} \times P_1 \times dR + \frac{1}{2} \times K_1 \times \kappa^*_1 \times P_1 \times (dR)^2 \\ &\qquad -K_2 \times D^*_{mod,2} \times P_2 \times dR + \frac{1}{2} \times K_2 \times \kappa^*_2 \times P_2 \times (dR)^2\,. \end{aligned}$$

We can now pull together the terms in dR and $(dR)^2$ to obtain:

$$d\Pi = -\left(D^*_{mod} \times P + K_1 \times D^*_{mod,1} \times P_1 + K_2 \times D^*_{mod,2} \times P_2\right) \times dR + \frac{1}{2} \times (\kappa^* \times P + K_1 \times \kappa^*_1 \times P_1 + K_2 \times \kappa^*_2 \times P_2) \times (dR)^2 .$$

Thus, in order for the portfolio to be immune to both small and larger changes, we have to chose K_1 and K_2 such that:

$$K_1 \times D^*_{mod,1} \times P_1 + K_2 \times D^*_{mod,2} \times P_2 = -D^*_{mod} \times P \quad \texttt{Delta Hedging}$$

$$K_1 \times \kappa^*_1 \times P_1 + K_2 \times \kappa^*_2 \times P_2 = -\kappa^* \times P \quad \texttt{Convexity Hedging.}$$

The solution of this system is:

$$K_1 = -\frac{P}{P_1} \times \left(\frac{D^*_{mod} \times \kappa^*_2 - D^*_{mod,2} \times \kappa^*}{D^*_{mod,1} \times \kappa^*_2 - D^*_{mod,2} \times \kappa^*_1}\right)$$

$$K_2 = -\frac{P}{P_2} \times \left(\frac{D^*_{mod} \times \kappa^*_1 - D^*_{mod,1} \times \kappa^*}{D^*_{mod,2} \times \kappa^*_1 - D^*_{mod,1} \times \kappa^*_2}\right) .$$

Exercise N° 10 (Exercise N° 08 and 09, continued; duration-convexity hedging).

(*i*) The modified duration and convexity of the short-maturity ZCB are given by $D^*_{zcb,1} = 2$ and $\kappa^*_{zcb,1} = 4$. In the case of the long-maturity one, we have $D^*_{zcb,2} = 10$ and $\kappa^*_{zcb,2} = 100$.

(*ii*) $K_1 = -0.4562$ and $K_2 = -1.1737$. This means that, to hedge against both small and large changes in interest rates, the corporation must short 0.4562 units of the 2-year ZCB, and 1.1737 units of the 10-year ZCB.

(*iii*) The following **Table 3** illustrates the performance of the hedging strategy under the three scenarios presented and analyzed in Exercise N° 08:

Yield Curve Shift	$CB(0,T)$	$B(0,T_2)$	$B(0,T_1)$	K_1	K_2	$d\Pi$
Initial Values	103.58	63.76	91.39	-0.4562	-1.1737	
$dR = 0.1\%$	102.75	63.13	91.21	-0.4562	-1.1737	0.0000
$dR = 1\%$	95.63	57.69	89.58	-0.4562	-1.1737	0.0003
$dR = 2\%$	88.38	52.20	87.81	-0.4562	-1.1737	0.0023
$dR = -0.1\%$	104.41	64.40	91.58	-0.4562	-1.1737	0.0000
$dR = -1\%$	112.29	70.47	93.24	-0.4562	-1.1737	-0.0003
$dR = -2\%$	121.84	77.88	95.12	-0.4562	-1.1737	-0.0027

As we would expect, the changes of the hedged portfolio are much smaller than under duration hedging, even for relatively large variation in the term structure.

Exercise N° 11.

Given that we have:

$$\begin{aligned} B(t,t+h) &= G_0(h) \text{ for } h \in [h_0,h_1)\,, \\ &= G_0(h)+G_1(h) \text{ for } h \in [h_1,h_2)\,, \\ &= G_0(h)+G_1(h)+\ldots+G_j(h) \text{ for } h \in [h_j,h_{j+1})\,, \\ &\text{etc.} \end{aligned}$$

For $h \in [0,h_1)$, we find:

$$\begin{aligned} &B(t,t+h) = \alpha_0+\beta_0 h+\gamma_0 h^2+\delta_0 h^3\,, \\ &\text{and since } B(t,t)=1 \;\Rightarrow\; \alpha_0 = 1\,, \\ &\Rightarrow\;\; B(t,t+h) := B_0(t,t+h) = 1+\beta_0 h+\gamma_0 h^2+\delta_0 h^3\,. \end{aligned} \tag{1}$$

For $h \in [h_1,h_2)$, we therefore can write:

$$\begin{aligned} B(t,t+h) := B_1(t,t+h) = (1+\beta_0 h+\gamma_0 h^2+\delta_0 h^3) \\ +[\alpha_1+\beta_1(h-h_1)+\gamma_1(h-h_1)^2+\delta_1(h-h_1)^3]\,. \end{aligned} \tag{2}$$

If we apply condition $i)$ to relations (1) and (2) we respectively find:

$$\begin{aligned} &\lim_{h\to h_1^-} B_0(t,t+h) = 1+\beta_0 h_1+\gamma_0 h_1^2+\delta_0 h_1^3\,, \\ &\lim_{h\to h_1^+} B_1(t,t+h) = (1+\beta_0 h_1+\gamma_0 h_1^2+\delta_0 h_1^3)+\alpha_1\,, \\ &B_1(t,t+h_1) = (1+\beta_0 h_1+\gamma_0 h_1^2+\delta_0 h_1^3)+\alpha_1\,, \\ &\text{thus, condition } i) \text{ implies : } \alpha_1 = 0\,. \end{aligned}$$

Now if we differentiate $B_0(t,t+h)$ and $B_1(t,t+h)$ we respectively find:

$$\begin{aligned} &B_0(t,t+h)' = \beta_0+2\gamma_0 h+3\delta_0 h^2\,,\;\; h\in[0,h_1)\,, \\ &B_1(t,t+h)' = \beta_0+2\gamma_0 h+3\delta_0 h^2+\beta_1 \\ &\qquad\qquad +\,2\gamma_1(h-h_1)+3\delta_1(h-h_1)^2\,,\;\; h\in[h_1,h_2)\,. \end{aligned} \tag{3}$$

This means that condition $ii)$ is satisfied if and only if $\beta_1 = 0$. Now, if we differentiate again the two relations in (3) we find:

$$\begin{aligned} &B_0(t,t+h)'' = 2\gamma_0+6\delta_0 h\,,\;\; h\in[0,h_1)\,, \\ &B_1(t,t+h)'' = 2\gamma_0+6\delta_0 h+2\gamma_1+6\delta_1(h-h_1)\,,\;\; h\in[h_1,h_2)\,. \end{aligned} \tag{4}$$

Consequently, condition $iii)$ implies $\gamma_1 = 0$.

Exercise N° 12.

We have a $\mathbb{R}^p$-valued square-integrable random vector $X = (x_1, \dots, x_p)'$ with mean vector $\mu = \mathbb{E}[X]$ and variance-covariance matrix $\Sigma = \mathbb{V}[X]$.

$a)$ Given that the variance-covariance matrix Σ is symmetric and positive semi-definite, from the Jordan Decomposition Theorem, we can write:

$$\Sigma = \Gamma\Lambda\Gamma' = \sum_{j=1}^{p} \lambda_j \gamma_j \gamma_j' \,, \text{ where } \Lambda = diag(\lambda_1, \dots, \lambda_p)\,, \text{ (where } \lambda_j\text{'s are the eigenvalues of } \Sigma)\,,$$

$\Gamma = (\gamma_1, \dots, \gamma_p)$ is an orthogonal matrix ($\Gamma^{-1} = \Gamma'$, i.e. $\gamma_i'\gamma_j = 0 \; \forall i \neq j$

with $||\gamma_j|| = 1$) where the j^{th} column is the j^{th} eigenvector γ_j of Σ.

$b)$ Given that Σ is (not only symmetric but also) positive semi-definite, we have that $\lambda_j \geq 0$ $\forall j \in \{1, \dots, p\}$. Let us prove this property of semi-definite matrices. The proof is based, first of all, on the definition of *Quadratic Form*, *Definiteness of Quadratic Forms and Matrices.*

Quadratic Form : A Quadratic Form $Q(x)$ is built from a $(p \times p)$ symmetric matrix $\mathcal{A}$ and a vector $x \in \mathbb{R}^p$: $Q(x) = x'\mathcal{A}x = \sum_{i=1}^p \sum_{j=1}^p a_{ij}x_i x_j$.

Definiteness of Quadratic Forms and Matrices : A Quadratic Form $Q(x)$ is *positive definite* if $Q(x) > 0 \; \forall x \neq 0$. It is *positive semi-definite* if $Q(x) \geq 0 \; \forall x \neq 0$. A symmetric matrix $\mathcal{A}$ is called positive definite (semi-definite) if the corresponding quadratic form Q is positive definite (semi-definite). They are denoted $\mathcal{A} > 0$ and $\mathcal{A} \geq 0$, respectively.

Theorem [Quadratic Forms can always be diagonalized] : If $\mathcal{A}$ is symmetric and $Q(x) = x'\mathcal{A}x$ is the corresponding quadratic form, then there exist a transformation $x \mapsto y = \Gamma'x$ such that : $x'\mathcal{A}x = \sum_{i=1}^{p} \lambda_i y_i^2$. Indeed, from JD theorem we have $\mathcal{A} = \Gamma\Lambda\Gamma'$ and, assuming $y = \Gamma'x$, we have that $x'\mathcal{A}x = x'\Gamma\Lambda\Gamma'x = y'\Lambda y = \sum_{i=1}^p \lambda_i y_i^2$.

Theorem : The symmetric matrix $\mathcal{A}$ is positive definite if and only if $\lambda_i > 0 \; \forall i \in \{1, \dots, p\}$; $\mathcal{A}$ is positive semi-definite if and only if $\lambda_i \geq 0 \; \forall i \in \{1, \dots, p\}$.

Proof (of positive semi-definite): $0 \leq \lambda_1 y_1^2 + \dots + \lambda_p y_p^2 = x'\mathcal{A}x$ for all $x \neq 0$ by the above mentioned theorem about the diagonalization of quadratic forms.

$c)$ The Principal Component transform of X is given by $Y = \Gamma'(X - \mu)$. This means that : $\mathbb{E}[Y] = 0$ and $\mathbb{V}[Y] = \Gamma'\Sigma\Gamma = \Gamma'(\Gamma\Lambda\Gamma')\Gamma = \Lambda$. Given that Λ is a diagonal matrix having in the main diagonal the p eigenvalues of Σ, namely $(\lambda_1, \dots, \lambda_p)'$, we have that each principal component Y_j has variance $\mathbb{V}[Y_j] = \lambda_j$ and it is uncorrelated with the others.

$d)$ The proof of this result is based on the following :

Theorem [Maximizing Quadratic Forms under Constraints] : If $\mathcal{A}$ and $\mathcal{B}$ are symmetric and $\mathcal{B} > 0$, then the maximum of $x'\mathcal{A}x$ under the constraints $x'\mathcal{B}x = 1$ is given by the largest eigenvalue of $\mathcal{B}^{-1}\mathcal{A}$. More generally:

$$\max_{\{x\,:\,x'\mathcal{B}x=1\}} x'\mathcal{A}x = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p = \min_{\{x\,:\,x'\mathcal{B}x=1\}} x'\mathcal{A}x\,,$$

where $(\lambda_1, \ldots, \lambda_p)$ are the eigenvalues of $\mathcal{B}^{-1}\mathcal{A}$. The vector which maximizes (minimizes) $x'\mathcal{A}x$ under the constraints $x'\mathcal{B}x = 1$ is the eigenvector of $\mathcal{B}^{-1}\mathcal{A}$ which corresponds to the largest (smallest) eigenvalue of $\mathcal{B}^{-1}\mathcal{A}$.

Now, we have to determine:

$$\begin{aligned}\max_{\{\delta:||\delta||=1\}} \mathbb{V}[\delta' X] &= \max_{\{\delta:||\delta||=1\}} \delta'\mathbb{V}[X]\delta \\ &= \max_{\{\delta:||\delta||=1\}} \delta'\Sigma\delta\,.\end{aligned}$$

The solution of this problem is obtained from the above mentioned theorem denoting $x = \delta$ and assuming $\mathcal{A} = \Sigma$ and $\mathcal{B} = I_p$, where I_p denotes the $(p \times p)$ identity matrix. We immediately obtain:

$$\max_{\{\delta:\delta'\delta=1\}} \delta'\Sigma\delta = \lambda_1\,,\ \text{ where } \lambda_1 \text{ is the}$$

$$\text{largest eigenvalue of } \mathcal{B}^{-1}\mathcal{A} = \Sigma.$$

The vector who maximizes $\delta'\Sigma\delta$ is therefore the eigenvector of Σ which corresponds to λ_1, that is γ_1. We have therefore:

$$\max_{\{\delta:||\delta||=1\}} \mathbb{V}[\delta' X] = \mathbb{V}[\gamma_1' X] = \gamma_1'\Sigma\gamma_1 = \lambda_1.$$

The last equality is also deduced from the JD of Σ: $\Sigma = \Gamma\Lambda\Gamma' \Rightarrow \Gamma'\Sigma\Gamma = \Lambda$ (remember that $\Gamma' = \Gamma^{-1}$).

Exercise N° 13.

We have that:

$$f(t, t+h) = B_0(h) + B_1(h) + B_2(h)\,,\ \text{ where } B_0(h) = \beta_0\,,\ \ B_1(h) = \beta_1\, e^{(-h\,/\,\theta)}\,,$$

$$B_2(h) = \beta_2\, \frac{h}{\theta}\, e^{(-h\,/\,\theta)}\,,\ \ \theta > 0\,.$$

a) The role of $B_0(h) = \beta_0$ (assuming $\beta_0 \neq 0$) - Given $\theta > 0$ we have:

$$\lim_{h\to\infty} f(t, t+h) := f(t, t+\infty) = \beta_0 = B_0(h)\,.$$

This means that, as far as h increases toward $+\infty$, the term $B_0(h) = \beta_0$ participate in the determination of the rate with increasing weight. Indeed, for h infinitely big, the (so called) long-term forward rate is β_0. For this reason we say that $B_0(h) = \beta_0$ determines mostly the forward rates with large residual maturity.

b) The role of $B_1(h) = \beta_1 \, e^{(-h\,/\,\theta)}$ (assuming $\beta_1 \neq 0$) - We have that $B_1(h)$ is a monotonic function. Indeed:

$$B_1(h)|_{h=0} = \beta_1\,,\ \ \lim_{h\to+\infty} B_1(h) = 0\,,$$

$$\frac{dB_1(h)}{dh} = -\frac{\beta_1}{\theta}e^{-h/\theta} \neq 0 \ \ \forall \ \ h > 0\,.$$

From the last relation we deduce that $B_1(h)$ is monotone decreasing (increasing) if $\beta_1 > 0$ ($\beta_1 < 0$). In any case, the weight of $B_1(h)$ in determining $f(t, t + h)$ is mostly important when h is close to zero: this means that $B_1(h)$ mostly affects short-term forward rates.

c) The role of $B_2(h) = \beta_2 \, \frac{h}{\theta} \, e^{(-h\,/\,\theta)}$ (assuming $\beta_2 \neq 0$) - In this case, the function $B_2(h)$ is not monotonic given that:

$$B_2(h)|_{h=0} = 0\,,\ \ \text{and}\ \ \lim_{h\to+\infty} B_2(h) = 0\,;$$

$$\frac{dB_2(h)}{dh} = \frac{1}{\theta}e^{-h/\theta}\left[\beta_2\left(1 - \frac{h}{\theta}\right)\right]\ \ \forall h > 0\,;\ \ \frac{dB_2(h)}{dh} = 0 \ \ \Leftrightarrow \ \ h = \theta\,;$$

This means that we are always able to find a residual maturity h (that equal to the positive number θ) such that $B_2(h)' = 0$. Consequently, the weight of $B_2(h)$ in determining $f(t, t+h)$ is most important for time to maturities around θ. For that reason, we says that it mostly affects, compared to $B_1(h)$, the medium-term forward rates.

d) Interpretation of β_0 - The constant term $B_0(h) = \beta_0$ determines the level around which the curve flatten when $h \to \infty$. This means that, not only it mostly determines long-term rates, but also constitutes a level for the entire term structure of forward rates. For this reason, β_0 is seen as a *level parameter*.

e) Interpretation of β_1 - We have that $\lim_{h\to0} f(t, t + h) = f(t, t) = r(t) = \beta_0 + \beta_1$. Now, given $f(t, t+\infty) = \beta_0$, we have that β_1 measures the difference between the very short and very long part of the term structure of forward rates. For that reason, β_1 is seen as a *slope parameter*.

f) Interpretation of β_2 - We easily observe that:

$$\frac{dB_2(h)}{dh} > 0 \ \text{iff} \ \ \beta_2 > 0 \ \ \text{and} \ \ h < \theta\,;\text{or} \ \ \beta_2 < 0 \ \ \text{and} \ \ h > \theta\,.$$

Thus, if $\beta_2 > 0$, $B_2(h)$ is positive and it is first increasing (for $h < \theta$) and then decreasing (for $h > \theta$) toward zero. Otherwise, if $\beta_2 < 0$, $B_2(h)$ is negative and it is first decreasing (for $h < \theta$) and then increasing (for $h > \theta$) toward zero. This means that, in the single non-monotonic term, namely $B_2(h)$, β_2 determines if there is an upward or downward "hump". For that reason, we can see β_2 as a *curvature parameter*.