Fixed Income and Credit Risk : exercise sheet n° 02

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Exercise N° 01.

Calculate the duration D of a bond paying (annually) a 8% annual coupon rate for 5 years (residual maturity), and the principal of 100 at the end of the fifth year. The price today (date $t = 0$) of the bond is denoted $CB(0, T)$. The yield to maturity (or the interest rate, assuming a flat term structure) is 10%.

Exercise N° 02 (Exercise N° 01, continued).

The interest rate increases from 10% to 10.1%. What is the resulting bond price change? Compare the *true price variation* with the one obtained from the Duration of the bond. Now, let us imagine that the interest rate moves from 10% to 15%. Compare again the true price variation with the one deduced from the bond Duration. What can we say about the information provided by the Duration about the bond price variations?

Exercise N° 03.

Calculate the Duration at date $t = 0$ of a perpetual bond delivering annually a coupon payment C and characterized by a YTM of Y.

Exercise N° 04.

Interest rates are at a 5% level. What is the Duration at date $t = 0$ of: *i*) a zero-coupon bond with 7-year to maturity and face value 100? *ii*) a perpetuity paying a yearly coupon rate of 8%? *iii*) a 10-year coupon bond paying a 5% coupon rate with face value 100?

Exercise N° 05.

Calculate the duration D and the convexity κ of a bond paying (annually) a 10% annual coupon rate for 5 years (residual maturity), and the principal of 100 at the end of the fifth year. The price today (date $t = 0$) of the bond is denoted $CB(0, T)$. The yield to maturity (or the interest rate, assuming a flat term structure) is 10%.

Exercise N° 06 (Exercise N° 05, continued).

Let us consider the bond described in the previous exercise. Accounting for both the duration and the convexity effects, what is the new bond price if interest rates rise from 10% to 11%?

Exercise N° 07.

Let us consider a continuously compounded flat term structure at a given level R. The modified duration and convexity at date t of a ZCB maturing at T are given by:

$$D^*_{mod,zcb} := -\frac{1}{B(t,T)}\frac{dB(t,T)}{d\,R} = T - t\,,$$

$$\kappa^*_{zcb} := \frac{1}{B(t,T)}\frac{d^2B(t,T)}{d\,R^2} = (T-t)^2\,,$$

being $B(t,T) = \exp[-R\,(T-t)]$. The date t duration and convexity of a coupon bond with semiannual n coupon payments, at dates $(T_1,\ldots,T_n = T)$, at an annual coupon rate c and unitary face value are given by (assuming $t < T_1$):

$$D^*_{mod,cb} := \sum_{i=1}^{n} w_i\,(T_i - t)\,,\quad \kappa^*_{cb} := \sum_{i=1}^{n} w_i\,(T_i-t)^2\,,$$

$$\texttt{where}\ w_i = \frac{(c/2)\,\times\,B(t,T_i)}{CB(t,T)}\,,\ \texttt{for}\ i = 1,\ldots,n-1\,;$$

$$w_n = \frac{(1+c/2)\,\times\,B(t,T_i)}{CB(t,T)}\,.$$

Let us assume now that $t = 0$, $R = 4.5\%$ and let us consider a coupon bond that matures in 10 years with semiannual coupon payments, at an annual coupon rate of 5%, and with face value $C_T = 100$. Determine its modified duration and convexity.

Exercise N° 08 (Exercise N° 07, continued; duration hedging strategy).

Let assume now that a corporation buys \$100 million (par value) of the 10-year coupon bond mentioned in the previous exercise (thus, implying a position of \$103.50 million) but it is worried about the losses that its portfolio may suffer from an upward shift in the term structure of interest rates. In order to manage this interest rate risk, the corporation decides to implement the following *duration hedging strategy*: entering into a position of K 10-year ZCBs $B(0,T)$ such that the value of the new portfolio thus created be less sensitive to interest rates variations. From the Table 1 of exercise 7 (solutions) the value of this ZCB is $B(0,T) = 100 \times 0.6376 = \$63.76$. Its duration is clearly $D^*_{mod,zcb} = 10$. Denoting by $CB(0,T) = \$103.58$ the value of the coupon bond, what is the position K in the ZCB such that the portfolio $\Pi = CB(0,T) + K \times B(0,T)$ is insensitive to a small parallel shift of the yield curve ? Compare the cases of a small shift with a change $dR =$ 10 basis points, a medium shift of $dR = 1\%$ and a large one of $dR = 2\%$. How does the duration hedge perform under these three scenarios? Interpret the results.

Exercise N° 09 (Exercise N° 08, continued; duration-convexity hedging strategy).

The previous exercise shows that the hedged portfolio loses money both when interest rates increases and decreases if the size of these shifts is large. This behavior of the hedged portfolio is due to the fact that the hedged portfolio is not *convexity hedged*. Indeed:

$$
\begin{aligned}
d\Pi &= dCB(0,T) + K \times dB(0,T) \\
&= -D^*_{mod,cb} \times CB(0,T) \times dR + \frac{1}{2} CB(0,T) \times \kappa^*_{cb} \times (dR)^2 \\
&\quad + K \times \left(-D^*_{mod,zcb} \times B(0,T) \times dR + \frac{1}{2} B(0,T) \times \kappa^*_{zcb} \times (dR)^2 \right) .
\end{aligned}
$$

Collecting the terms in dR and $(dR)^2$ we obtain:

$$
\begin{aligned}
d\Pi &= -\left(D^*_{mod,cb} \times CB(0,T) + K \times D^*_{mod,zcb} \times B(0,T) \right) \times dR \\
&\quad + \frac{1}{2} \times \left(CB(0,T) \times k^*_{cb} + K \times B(0,T) \times k^*_{zcb} \right) \times (dR)^2 .
\end{aligned}
$$

Duration hedging eliminates the first parenthesis in the latter equation, however the second one is generally not zero. In particular, if the parenthesis is negative, then the duration hedging strategy tends to generate a loss irrespective of a positive or negative interest rate change, given that $(dR)^2 > 0$.

One strategy to hedge against both small and large interest rate variations is two use two securities to hedge both duration and convexity simultaneously. More specifically, let P_1 and P_2 be the prices of two securities (such as short-term and long-term ZCBs), with $D^*_{mod,1}$, $D^*_{mod,2}$, κ^*_1 and κ^*_2 their durations and convexities. Determine the positions K_1 and K_2 in the two assets such that the portfolio value $V = P + K_1 \times P_1 + K_2 \times P_2$ is hedged against both small yield changes (small dR) and large yield changes (large dR and thus large $(dR)^2$).

Exercise N° 10 (Exercise N° 08 and 09, continued; duration-convexity hedging).

Let us consider again Exercise N° 08, but assume that in addition to the ZCB with maturity $T_2 = 10$ years, the corporation also uses a ZCB with a short maturity of $T_1 = 2$ years. From Table 1 of Exercise N° 07 (solution) we have that the price of the 2-year ZCB is $B(0,T_1) = \$91.39$. From the same table, the convexity of the bond we want to hedge is $\kappa^*_{cb} = 73.87$.

(*i*) Determine the modified duration and convexities of the 2-year and 10-year ZCBs.

(*ii*) On the basis of the formulas determined in the previous exercise, calculate the amounts K_1 and K_2 invested in the short-maturity and long-maturity ZCBs, respectively, such that the portfolio is *duration-convexity hedged.*

(*iii*) Does this hedging strategy work any better than the simpler duration hedging strategy?

Exercise N° 11.

Let us consider the spline approximation of the discount function $B(t, t+h)$:

$$B(t,t+h) = \sum_{j=0}^{k-1} G_j(h)\mathbb{I}_j(h)\,, \text{where } \mathbb{I}_j(h) = \begin{cases} 1 \text{ if } h \geq h_j \\ \\ 0 \text{ otherwise}\,, \end{cases}$$

$$\text{and where } G_j(h) = \alpha_j + \beta_j(h-h_j) + \gamma_j(h-h_j)^2 + \delta_j(h-h_j)^3\,,$$

$$\text{with } \alpha_j\,, \beta_j\,, \gamma_j \text{ and } \delta_j \text{ unknown parameters}\,.$$

To guarantee the continuity and the smoothness of the spline approximation, when we move from $h \in [0, h_1)$ to $h \in [h_1, h_2)$, we impose the following three conditions:

$$\begin{aligned} i)\ \lim_{h\to h_1^-} B(t,t+h) &= \lim_{h\to h_1^+} B(t,t+h) = B(t,t+h_1) \\ ii)\ \lim_{h\to h_1^-} B'(t,t+h) &= \lim_{h\to h_1^+} B'(t,t+h) < \infty \\ iii)\ \lim_{h\to h_1^-} B''(t,t+h) &= \lim_{h\to h_1^+} B''(t,t+h) < \infty\,. \end{aligned}$$

Show these three conditions (along with the condition $B(t,t) = 1$) imply: $\alpha_0 = 1$, $\alpha_1 = \beta_1 = \gamma_1 = 0$.

Exercise N° 12.

Let us consider a $\mathbb{R}^p$-valued square-integrable random vector $X = (x_1, \ldots, x_p)'$ with mean vector $\mu = \mathbb{E}[X]$ and variance-covariance matrix $\Sigma = \mathbb{V}[X]$.

a) Using the Jordan Decomposition Theorem, how can I write Σ ?

b) What can I say about the eigenvalues of Σ ? Prove the result.

c) How the principal component transform of X is defined? If we denote with Y the principal component transform, which are its mean vector and variance-covariance matrix? In particular, which is the variance of each principal component?

d) Now, let us assume to order the p eigenvalues of Σ from the largest to the smallest: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$. Show that the first principal component of X, Y_1 (say), has maximal variance among all possible standardized linear combinations of X. In other words, show that:

$$\mathbb{V}[\gamma_1' X] = \max_{\{\delta:||\delta||=1\}} \mathbb{V}[\delta' X]\,.$$

Exercise N° 13.

Nelson and Siegel (1987) specify the instantaneous forward rate curve in the following way:

$$\begin{aligned} f(t,t+h) &= B_0(h) + B_1(h) + B_2(h)\,, \ \text{ where} \\ B_0(h) &= \beta_0\,, \\ B_1(h) &= \beta_1\, e^{(-h\,/\,\theta)}\,, \\ B_2(h) &= \beta_2\, \frac{h}{\theta}\, e^{(-h\,/\,\theta)}\,, \\ \theta > 0\,. \end{aligned}$$

Describe the role of the three components $B_0(h)$, $B_1(h)$ and $B_2(h)$ and provide an interpretation of the associated parameters β_0, β_1 and β_2. Moreover, explain why θ is seen as a scale or location parameter.