Fixed Income and Credit Risk : solutions for exercise sheet n° 01

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Exercise N° 01.

The data of the problem are : $P_t = 25000$, $R = 3\%$, $n = 5$. If the annual rate R is compounded quarterly ($m = 4$), the value of the principal after 5 years is:

$$V_F(4,5) = 25000 \times (1 + 0.03/4)^{4\times 5} = 29029.60\,,$$

the effective annual rate is $R^{eff} = (1+0.03/4)^4 - 1 = 0.03034$ and the equivalent annual continuously compounded rate is $R_\infty = 4\ln(1 + 0.03/4) = 0.02989$. If the annual rate is compounded monthly ($m = 12$), then:

$$V_F(12,5) = 25000 \times (1 + 0.03/12)^{12\times 5} = 29040.42\,,$$

the effective annual rate is equal to $R^{eff} = (1+0.03/12)^{12} - 1 = 0.03042$ and the equivalent annual continuously compounded rate is $R_\infty = 12\ln(1 + 0.03/12) = 0.02996$.

Exercise N° 02.

The coupon bond price is:

$$\begin{aligned} CB(t,t+2) &= \tfrac{0.04\times 100}{2} \times [B(t,t+0.5) + B(t,t+1) + B(t,t+1.5)] \\ &\quad + \left(100 + \tfrac{0.04\times 100}{2}\right) \times B(t,t+2) \\ &= 2.0 \times [0.99 + 0.985 + 0.98] + 102 \times 0.975 \\ &= 105.36\,. \end{aligned}$$

Exercise N° 03.

Given at date t a coupon bond with maturity date T, price $CB(t,T)$ and payments $\{C_1,\ldots,C_n\}$ at dates $\{T_1,\ldots,T_n\}$, respectively, the associated annually compounded yield to maturity is the value $Y^{CB}(t,T)$ satisfying the equation:

$$CB(t,T) = \sum_{i=1}^{n} C_i (1 + Y^{CB}(t,T))^{-(T_i - t)}\,.$$

Here we have $T - t = 3$ years ($n = 3$), $Y^{CB}(t,T) = 5\%, 7\%, 9\%, 10\%$ (annually compounded), $C_1 = C_2 = 0.1 \times 100$ and $C_3 = (0.1 + 1) \times 100$. Applying the formula to the data of the problem we have that:

$$
\begin{aligned}
CB(t,t+3) &= 113.616 \text{ if } Y^{CB}(t,t+3) = 5\% \\
&= 107.873 \text{ if } Y^{CB}(t,t+3) = 7\% \\
&= 102.531 \text{ if } Y^{CB}(t,t+3) = 9\% \\
&= 100.000 \text{ if } Y^{CB}(t,t+3) = 10\% \, .
\end{aligned}
$$

Exercise N° 04.

The annual yield to maturity $Y^{CB}(t, t + 4y)$ solves the equation:

$$
CB(t,T) = \sum_{i=1}^{4} 7 \times (1 + Y^{CB}(t,T))^{-i} + \frac{100}{(1 + Y^{CB}(t,T))^4} \, .
$$

We find 6.13%.

Exercise N° 05.

From the definition of annually compounded yield to maturity $Y^{CB}(0,T) = Y$ and using the definition of geometric progression we have:

$$
\begin{aligned}
\frac{CB(0,T)}{C_T} &= \sum_{i=1}^{T} c \times (1+Y)^{-i} + \frac{1}{(1+Y)^T} \, , \\
&= c \times \frac{1}{Y} \times (1 - (1+Y)^{-T}) + (1+Y)^{-T} \, .
\end{aligned}
$$

If $c = Y$ then $CB(0,T) = C_T$ and the bond is said to be a *par bond*. If $c > Y$ then $CB(0,T) > C_T$ and the bond is called a *premium bond*. If $c < Y$ we have $CB(0,T) < C_T$ and the bond is called a *discount bond*.

Exercise N° 06.

a) We have the following five bonds:

Table 1

	Residual Maturity (years)	Annual Coupon Rate (%)	Price
Bond A	1	5.00	100.91
Bond B	2	6.00	103.02
Bond C	3	7.50	107.54
Bond D	4	5.25	101.18
Bond E	5	8.00	112.72

Then from the definition of annually compounded yield to maturity of a coupon bond we have, for instance, for Bond C :

$$
\begin{aligned}
CB_C(t,T) &= \sum_{i=1}^{n} C_i (1 + Y_C^{CB}(t,T))^{-(T_i - t)}\,, \\
&= \sum_{i=1}^{3} 7.50 \times (1 + Y_C^{CB}(t, t+3y))^{-i} + \frac{100}{(1 + Y_C^{CB}(t, t+3y))^3} \\
&\Rightarrow\ Y_C^{CB}(t, t+3y) = 4.745\%\,.
\end{aligned}
$$

Similarly, we find that $Y_A^{CB}(t, t+1y) = 4.05\%$, $Y_B^{CB}(t, t+2y) = 4.39\%$, $Y_D^{CB}(t, t+4y) = 4.92\%$ and $Y_E^{CB}(t, t+5y) = 5.06\%$.

b) The annually compounded discount rates (i.e., ZCB yield to maturities) $Y(t, t+i)$, $i \in \{1y, \dots, 5y\}$, are determined recursively as follows:

$$
\begin{aligned}
100.91 &= \frac{105}{1 + Y_A(t,t+1)} \quad \Rightarrow\ Y_A(t,t+1) = 4.05\% \\
103.02 &= \frac{6.00}{1 + Y_A(t,t+1)} + \frac{106}{(1 + Y_B(t,t+2))^2} \quad \Rightarrow\ Y_B(t,t+2) = 4.40\%\,, \\
107.54 &= \frac{7.50}{1 + Y_A(t,t+1)} + \frac{7.50}{(1 + Y_B(t,t+2))^2} + \frac{107.50}{(1 + Y_C(t,t+3))^3} \quad \Rightarrow\ Y_C(t,t+3) = 4.78\%\,, \\
101.18 &= \frac{5.25}{1 + Y_A(t,t+1)} + \frac{5.25}{(1 + Y_B(t,t+2))^2} + \frac{5.25}{(1 + Y_C(t,t+3))^3} + \frac{105.25}{(1 + Y_D(t,t+4))^4} \\
&\qquad \Rightarrow\ Y_D(t,t+4) = 4.95\%\,, \\
112.72 &= \frac{8.00}{1 + Y_A(t,t+1)} + \frac{8.00}{(1 + Y_B(t,t+2))^2} + \frac{8.00}{(1 + Y_C(t,t+3))^3} + \frac{8.00}{(1 + Y_D(t,t+4))^4} \\
&\qquad + \frac{108}{(1 + Y_E(t,t+5))^5} \quad \Rightarrow\ Y_E(t,t+5) = 5.12\%\,.
\end{aligned}
$$

c) Following the procedure at point *a*), we find $Y_{B'}^{CB}(t, t+2y) = 4.38\%$. Given the discount rates (determined at point *b*)) $Y_A(t,t+1) = 4.05\%$ and $Y_B(t,t+2) = 4.40\%$, the arbitrage-free B'-bond price is:

$$
\begin{aligned}
CB(t,T) &= \sum_{i=1}^{n} 1_{\{t<T_i\}} C_i B(t,T_i)\,, \\
&= \frac{12}{1.0405} + \frac{112}{(1.044)^2} = 114.29
\end{aligned}
$$

which is the market price, and therefore there are no-arbitrage opportunities on B'.

d) The ZCB-based transaction, inducing a risk-less profit, is the following:

- sell the (over-priced) bond B' at price $CB_{B'}(t,t+2y) = 115.00$.
- buy 12 ZCBs with one-year to maturity at the price $B(t,t+1)$.
- buy 112 ZCBs with two-year to maturity at the price $B(t,t+2)$.

The cash-flow table for the transaction is:

Table 2

Periods	t	$t+1y$	$t+2y$
Sell bond B'	115.00	-12.00	-112.00
Buy 12 ZCBs $B(t,t+1)$	-11.533	12.00	0.00
Buy 112 ZCBs $B(t,t+2)$	-102.758	0.00	112.00
Total	0.71	0.00	0.00

The risk-less profit from the transaction is 0.71.

e) One can execute at date t the following transaction:

- sell the (over-priced) bond B' at price $CB_{B'}(t,t+2y) = 115.00$.
- buy 112/106 (under-priced) bonds B.
- buy 5.66/105 bonds A.

The cash-flow table for the transaction is:

Table 3

Periods	t	$t+1y$	$t+2y$
Sell bond B'	115.00	-12.00	-112.00
Buy 112/106 bonds B	-108.85	6.34	112.00
Buy 5.66/105 bonds A	-5.44	5.66	0.00
Total	0.71	0.00	0.00

The risk-less profit from the transaction is 0.71.

Exercise N° 07.

The no-arbitrage price of the bullet bond is:

$$
\begin{aligned}
CB(t,t+3y) &= (0.07\cdot 100)\cdot 0.98 + (0.07\cdot 100)\cdot 0.94 \\
&\quad +[(0.07\cdot 100)+100]\cdot 0.90 = 109.74 \;<\; 112\,.
\end{aligned}
$$

In the bond market there is an arbitrage opportunity given that at date t I can buy 7 one-year, 7 two-year and 107 three-year ZCBs (earning 109.74), and selling the bullet bond at the market price $CB(t, t+3y) = 102$. You obtain a positive net profit at t of $112 - 109.74$. Indeed, the cash-flow table for the transaction is:

Table 4

Periods	t	$t+1y$	$t+2y$	$t+3y$
Sell $CB(t, t+3y)$	112.00	-7.00	-7.00	-107.00
Buy 7 ZCBs $B(t, t+1)$	-6.86	7.00	0.00	0.00
Buy 7 ZCBs $B(t, t+2)$	-6.58	0.00	7.00	0.00
Buy 107 ZCBs $B(t, t+3)$	-96.3	0.00	0.00	107.00
Total	2.26	0.00	0.00	0.00

Exercise N° 08.

We know that the continuously compounded forward rate prevailing at date t, for the period $[\tau, T]$, is the rate $R(t, \tau, T)$ such that:

$$\exp[-R(t,T) \times (T-t)] = \exp[-R(t,\tau) \times (\tau - t)] \times \exp[-R(t,\tau,T) \times (T-\tau)]\,,$$

that is, $B(t,T) = B(t,\tau) \times \exp[-R(t,\tau,T) \times (T-\tau)]$. Now, let us create at date t the following portfolio: buy 1 forward contract (value zero at t by definition), buy 1 unit of the ZCB with price $B(t,\tau)$ and sell $B(t,\tau)/B(t,T)$ units of the ZCB with price $B(t,T)$.
The cash-flow table for the transaction is:

Table 5

Periods	t	τ	T
buy 1 forward contract	0	-1	$e^{[(T-\tau)R(t,\tau,T)]}$
buy ZCB $B(t,\tau)$	- $B(t,\tau)$	1	0
sell $\frac{B(t,\tau)}{B(t,T)}$ ZCBs $B(t,T)$	$B(t,\tau)$	0	-$\frac{B(t,\tau)}{B(t,T)}$
Total	0.00	0.00	$e^{[(T-\tau)R(t,\tau,T)]} - \frac{B(t,\tau)}{B(t,T)}$

This portfolio, with value of zero at t, produce a net cash-flow of zero at τ and a positive cash-flow of $e^{[(T-\tau)R(t,\tau,T)]} - \frac{B(t,\tau)}{B(t,T)} > 0$ at T. This is, by definition, an arbitrage.

Exercise N° 09.

The continuously compounded forward rate is :

$$R(t, t+2, t+5) = -\frac{1}{3}\ln(93/97) = 0.0140\,,$$

and the annually compounded one is:

$$Y(t, t+2, t+5) = \left(\frac{97}{93}\right)^{1/3} - 1 = 0.0141\,.$$

Exercise N° 10.

The indexed principal was $C_T^* = 1000000 \times \frac{135}{120} = 1125000$. The coupon payment was then $C_1^* = 1125000 \times \frac{0.035}{2} = 19687.50$, instead of $C_1 = 1000000 \times \frac{0.035}{2} = 17500.00$ in the case of a fixed payment in nominal terms. The difference $(C_1^* - C_1)$ is the compensation for inflation risk.

Exercise N° 11.

The accrued interest of the bond should be added to the clean price in order to determine the dirty price (also called invoice price), which will be the amount that the trader will have to finance overnight. It is given by:

$$AI = \frac{6.375}{2} \times \frac{88}{181} = 1.5497\,.$$

The dirty price (to be financed in the repo market) is $CB = 118.8420 + 1.5497 = 120.3917$. On a 10 million dollar par, the amount to be financed is $12,039,170$ dollars. To make sure that the repo dealer earns a repo rate of $r = 6\%$, we can compute the amount of cash that the trader must pay the repo dealer on May 15, 2007, as follows:

$$12,039,170 \times \left(1 + \frac{0.06}{360}\right) = 12,041,178.90\,.$$

On May 15, 2007, the dirty price of the bond (including an additional day we have an accrued interest of 1.5673) is 120.4093 and thus the trader, selling 10 million dollar par in the market, earns $12,040,930$ dollars with an associated profit of $-245,479$ dollars.

If the coupon rate is 9% (annual basis) instead of 6.375%, we have that the new accrued interest is:

$$AI = \frac{9.00}{2} \times \frac{88}{181} = 2.1878\,.$$

and thus the dirty price is $CB = 121,0298$. On a 10 million dollar par, the amount to be financed is $12,102,980$ dollars. The amount to be paid to the repo dealer is now $12,105,000$, the dirty price on May 15, 2007, is 121.0547 and thus the trader, selling the bond, earns $12,105,471$ dollars with an associated profit of (around) 470 dollars.

Exercise N° 12.

The profit $\Pi(t,T)$ (say) of the trader is:

$$
\begin{aligned}
\Pi(t,T) &= P_T - P_t - \left[P_t \times r \times \frac{n}{360}\right] \\
&= (CB_T^{clean} + AI_T) - (CB_t^{clean} + AI_t) \times \left[1 + r \times \frac{n}{360}\right] \\
&= CB_T - CB_t \times \left(1 + r \times \frac{n}{360}\right).
\end{aligned}
$$

It is clear that $\Pi(t,T) = 0$ if and only if

$$\frac{CB_T - CB_t}{CB_t} = r \times \frac{n}{360}$$

where the RHS of the equation is the appropriate repo rate for the time interval (t,T). If

$$\frac{CB_T - CB_t}{CB_t} > r \times \frac{n}{360}$$

the trade provides a positive carry, while if we have

$$\frac{CB_T - CB_t}{CB_t} < r \times \frac{n}{360}$$

the trade provides a negative carry.