Fixed Income and Credit Risk

Lecture 1

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Fixed Income and Credit Risk

Lecture 1

Fixed Income Markets and Securities

Outline of Lecture 1

1.3 <u>**Basics of Fixed Income Securities**</u>

1.3.1 <u>Time value of money</u>

1.3.2 <u>Discount Factors: Bonds</u>

1.3.3 <u>Bond market conventions</u>

1.3.4 <u>Bond yields and zero-coupon rates</u>

1.3.5 <u>Forward rates</u>

1.3.6 Forward Rates and Forward Discount Factors

1.3.7 Par yields

1.3.8 Inflation-indexed bonds

1.1 FIXED INCOME AND CREDIT RISK: INTRODUCTION

- The purpose of this course is to present "methods" (principle/models) to determine the "fair price" of fixed income securities with and without "credit risk" as well as fixed income derivatives and credit derivatives.

- What the class of fixed income securities includes ? It includes, for sure, securities where the issuer promises one or several FIXED, PREDETERMINED payments at given points in time.

  Examples : Zero-coupon bonds, Coupon bonds.

$\rightarrow$ when the issuer is a national government, the asset is (in general) assumed to be non-defaultable (absence of credit risk).

$\rightarrow$ bonds are also issued by other institutions like banks and companies: we have *corporate bonds* with a positive probability of default (credit risk!).

- We will also consider other financial assets in that class, even if their payoffs are not fixed and known at the time they are purchased (interest rate derivatives). Why ? Because that payoffs depend on the price of some "basic" fixed income security.

  Examples : Options and futures on bonds or interest rates, caps and floors, swaps and swaptions.

- The prices of fixed income securities are frequently expressed in terms of interest rates or yields $\Rightarrow$ it is important to understand what's behind the dynamics of interest rates in order to understand how to correctly price that assets.

- The fundamental concept in the analysis of fixed income securities is the **term structure of interest rates** (also called **yield curve**).

- The interest rate on a loan will depend in general on the **maturity date** of that loan $\Rightarrow$ differences between short-term and long-term interest rates.

- the term structure of interest rates defines the relationship between the interest rate and the maturity of the loan.

1.2 FIXED INCOME MARKETS AND SECURITIES

- Debt markets have expanded a lot in the last two decades. Now relevant issuers include:

a) **Treasury (Sovereign) Securities:** Issued by governments (USA, Japan, UK, etc.);

b) **Agency Securities:** Debt securities issued by government agencies, such as the Federal Home Loan Bank (FHLB), the Tennessee Valley Authority (TVA), the Federal National Mortgage Association (FNMA) and Government National Mortgage Association (GNMA).

$c)$ **Corporate Securities:** Debt securities issued by corporations (both investment grade and noninvestment grade).

$d)$ **Mortgage-Backed Securities:** Debt securities backed by pools of mortgages.

$e)$ **Asset-Backed Securities:** Securities backed by a portfolio of assets, such as credit card receivables.

$f)$ **Municipal issues:** Debt securities issued by state gov. and municipalities.

$g)$ **Derivative Securities:** Interest rate Swaps, Forwards, Options, Futures.

$\hookrightarrow$ *backed by* = collateralized by = guaranteed by ;

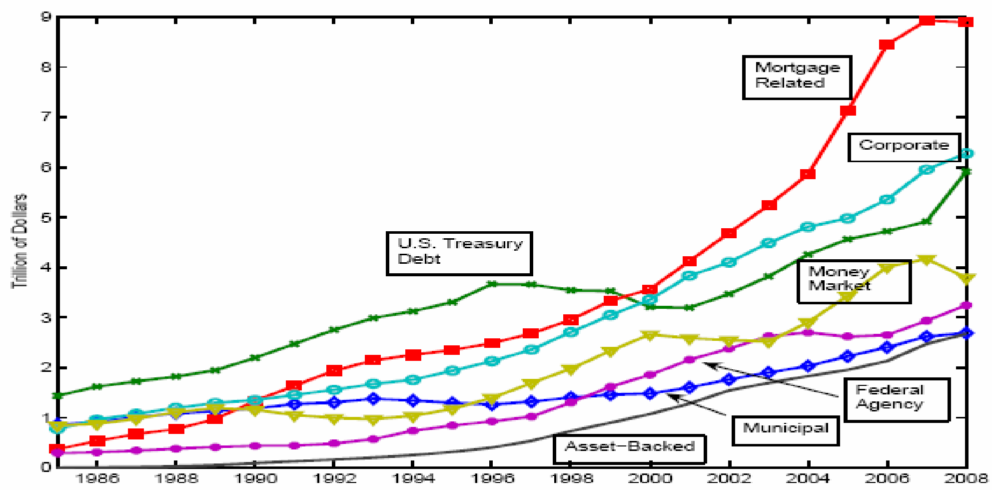
From Veronesi (2010, Chapter 1): Fixed Income Markets, Dec. 2008

Market	Market Value (billion of dollars)	Notional (billion of dollars)
U.S. Treasury Debt	5,912.2	
U.S. Municipal Debt	2,690.1	
U.S. Federal Agency Securities	3,247.4	
U.S. Money Market	3,791.1	
Mortgage-Backed Securities	8,897.3	
Asset-Backed Securities	2,671.8	
OTC Interest Rate Swaps	8,056.07*	356,771.58*
OTC Interest Rate Forwards	87.82*	39,370.22*
OTC Interest Rate Options	1,119.56*	62,162.19*
Exchange Traded Futures		19,271.05
Exchange Traded Options		35,161.34
U.S. Corporate Debt	6,280.6	
Credit Derivatives	3,172*	45,852*

Source: Securities Industry and Financial Market Association (SIFMA) and Bank for International Settlements (BIS).
* As of June 2008.

From Veronesi (2010, Chapter 1): Growth in Market Size

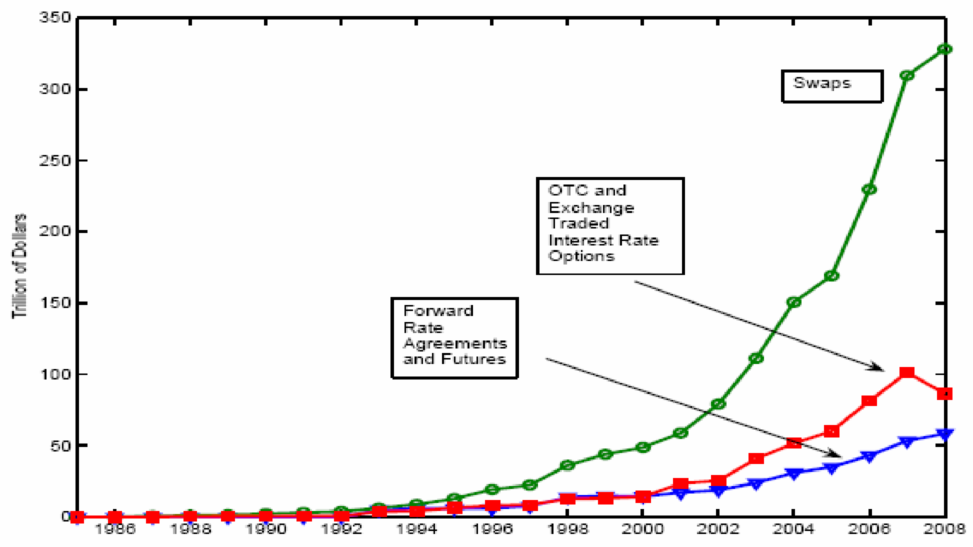
Figure 1.1 The Growth in Market Size



Source: The Securities Industry and Financial Markets Association (SIFMA)

From Veronesi (2010, Chapter 1): Growth in Derivatives Markets: Notional

Figure 1.2 The Growth in Derivatives Markets: Notional



Source: SIFMA and Bank for International Settlement

1.2.1 The Government Debt Markets

- The simplest fixed income securities are BONDS.

- A bond is a tradeble loan agreement in which the issuer sells a contract promising the holder a predetermined payment schedule.

- These payments may be fixed in nominal terms (a fixed-interest bond) or they may be linked to some index (an index-linked bonds) like the Consumer Price Index (CPI).

$\rightarrow$ We have :

- **Zero Coupon Bonds:** securities that only pay the principal at maturity (T-bills).

- **Fixed Rate Coupon Bonds:** securities that pay a fixed coupon over a given period (semiannually) plus the principal at maturity (T-notes and T-bonds).

- **Floating Rate Coupon Bonds:** securities that have the coupon indexed to some other short term interest rate, varying over time. The U.S. government does not issue floating rate bonds, other governments do. For instance, Italy issues the CCT bond, which is an Italian Treasury debt security whose coupon rate is indexed to the 6 month rate of Italian 6-month T-bills (BOT).

☐ **Treasury Inflation Protected Securities (TIPS):** securities with the principal indexed to inflation, so that coupon payments move accordingly.

- *Treasury bills, bonds and notes* (in the U.S. bond market), or *gilt-edged securities* (in the UK market) provide fixed nominal payments; *treasury inflation-protected securities* (TIPS; in the U.S. bond market) provide interest rate payments that rise with inflation and fall with deflation (measured by CPI).

i) Treasury bills (or T-bills) mature in one year or less, and they do not pay interest prior to maturity. Regular weekly T-bills are commonly issued with time-to-maturity of 28 days, 91 days, 182 days, and 364 days.

$ii)$ Treasury notes (or T-notes) mature in 2 to 10 years. They have a coupon payment every six months and are commonly issued with time-to-maturities of 2, 3, 5, 7 or 10 years.

$iii)$ Treasury bonds (or T-bonds) have the longest time-to-maturity, from 20 to 30 years, and they have a coupon payment every six months.

$iv)$ TIPS are currently offered in 5-year, 10-year and 20-years of time-to-maturity. They have a coupon payment every six months : the coupon rate is constant but generates a different amount of interest when multiplied by the inflation-adjusted principal (also named face value or par value).

- We also have:

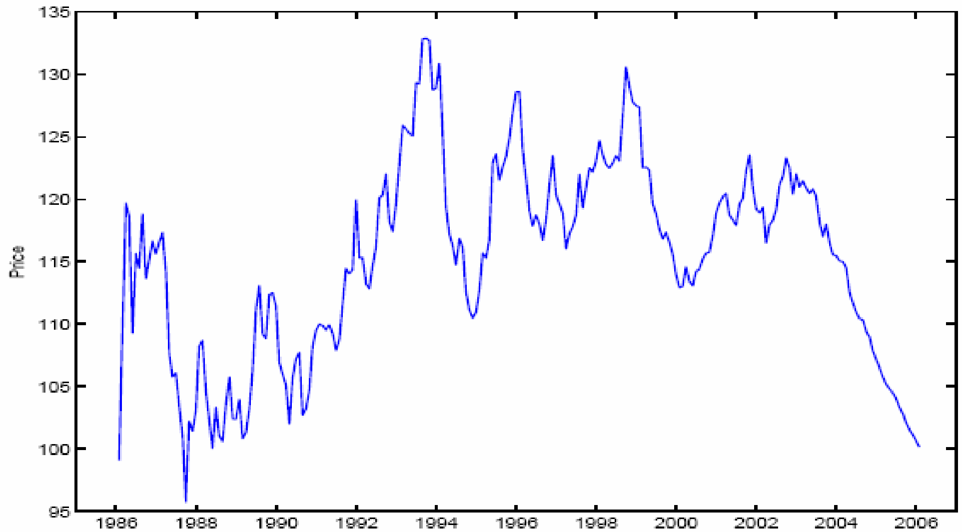
☐ **Separate Trading of Registered Interest and Principal of Securities (STRIPS):**

Artificial zero coupon bonds constructed by stripping off separate interest and principal payments from a coupon bond.

☐ **The Municipal Debt Market**

From Veronesi (2010, Chapter 1)

Figure 1.3 The February 2006, 9.375%, 20-Year Bond Price Path



Source: Center for Research in Security Prices

1.2.2 The Money Market

- Market for short-term borrowing and lending of banks and financial institutions:

☐ **Federal Funds (FF) Rate:** rate for borrowing / lending balances kept at the Federal Reserve[see Section 1.2.3].

☐ **EONIA (Euro Overnight Index Average):** it can be seen the Euro Area banking system equivalent of the effective Federal Funds rate [see Section 1.2.4].

☐ **LIBOR and EURIBOR:** average interest rate the banks charge to each other for short term uncollateralized borrowing / lending [see Section 1.2.4].

- ☐ **Repo Rate:** interest rate charged for short term borrowing / lending with collateral [see Section 1.2.5].

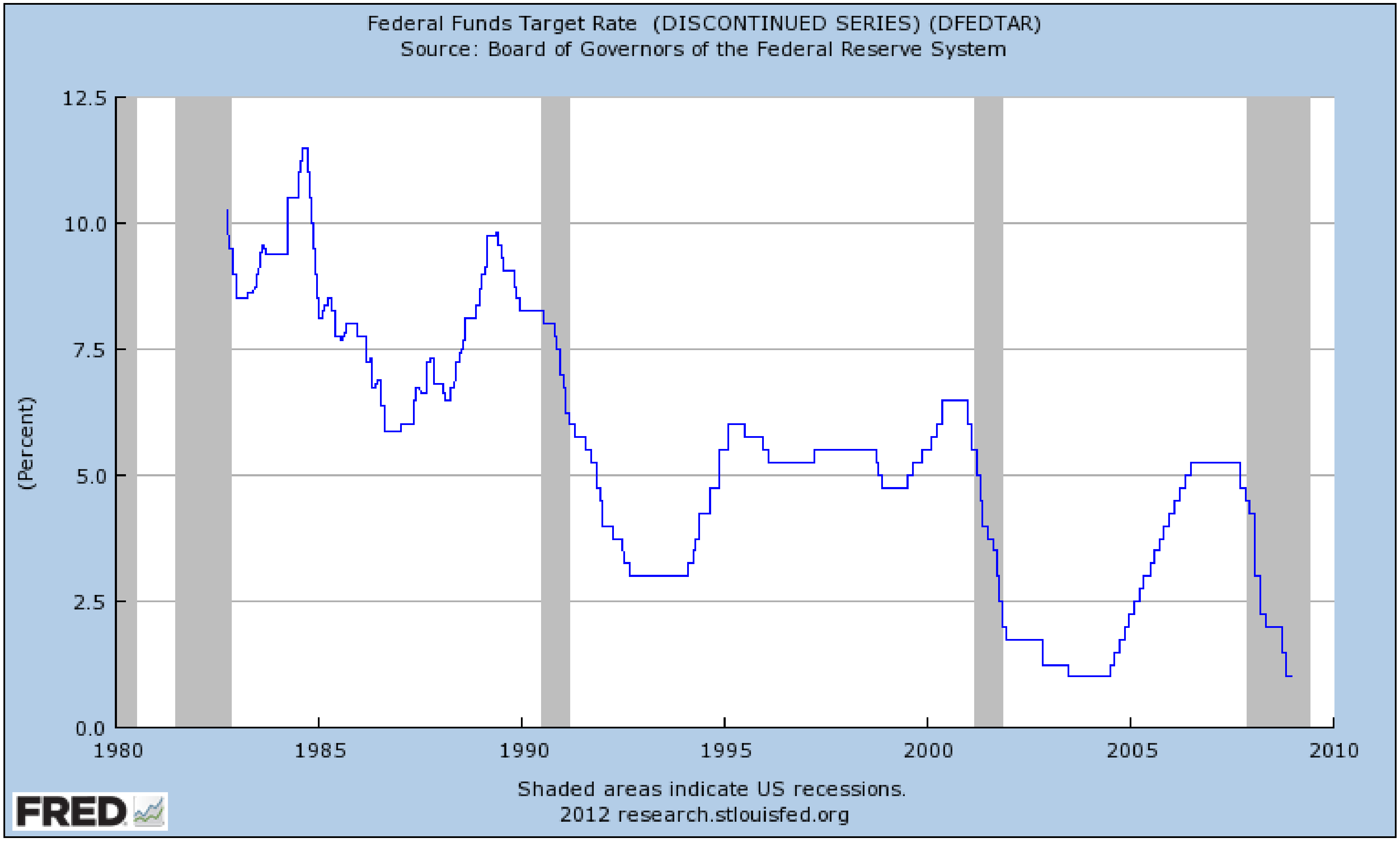
- ☐ **Eurodollar Rate:** it is the rate of interest on a dollar deposit in a European-based bank. These are short-term deposits from 3 months to one year. In particular, the 90-day Eurodollar rate has become a standard reference to gauge conditions of the interbank market (see following lectures). For instance:

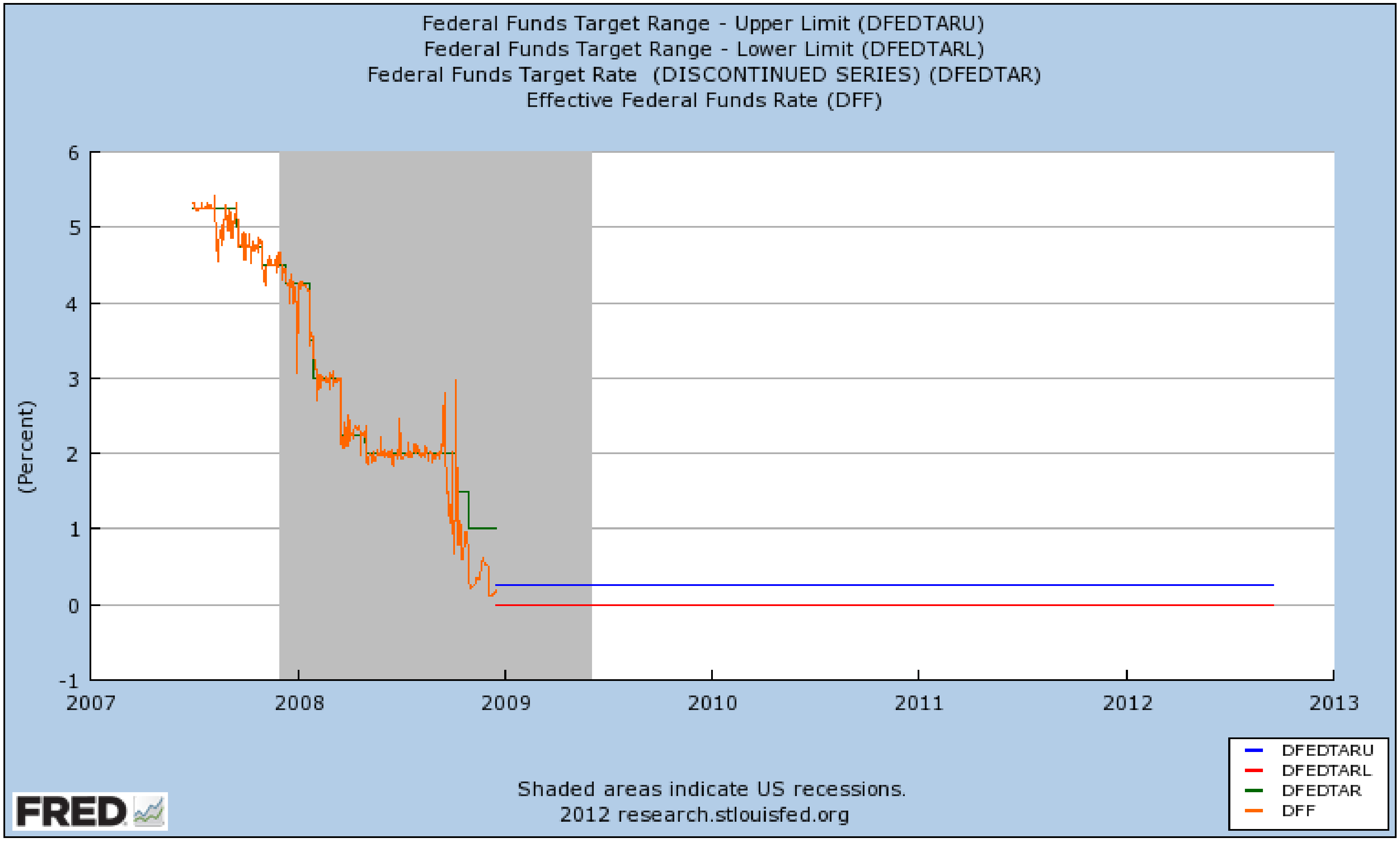
  – the market of Eurodollar futures and options (financial derivatives traded at the Chicago Mercantile Exchange that allow financial institutions to bet on or hedge against the future evolution of the Eurodollar rate) is among the largest and most liquid derivative markets in the world.

1.2.3 Federal Funds (FF) Rate and Eurosystem Rates

- In the U.S. banks and other institutions must keep some amount of capital (called federal funds) within the Federal Reserve (U.S. system of central banks).
  - Banks with a reserve surplus may then lend some of their reserves to banks with a reserve deficit, on a uncollateralized basis.
  - The **Federal Funds Rate** is the interest rate at which banks actively trade the federal funds with each other, usually overnight.
  - The **Effective Federal Funds (EFF) rate** is the volume-weighted average rate that banks charge to each other to lend or borrow reserves at the Fed.

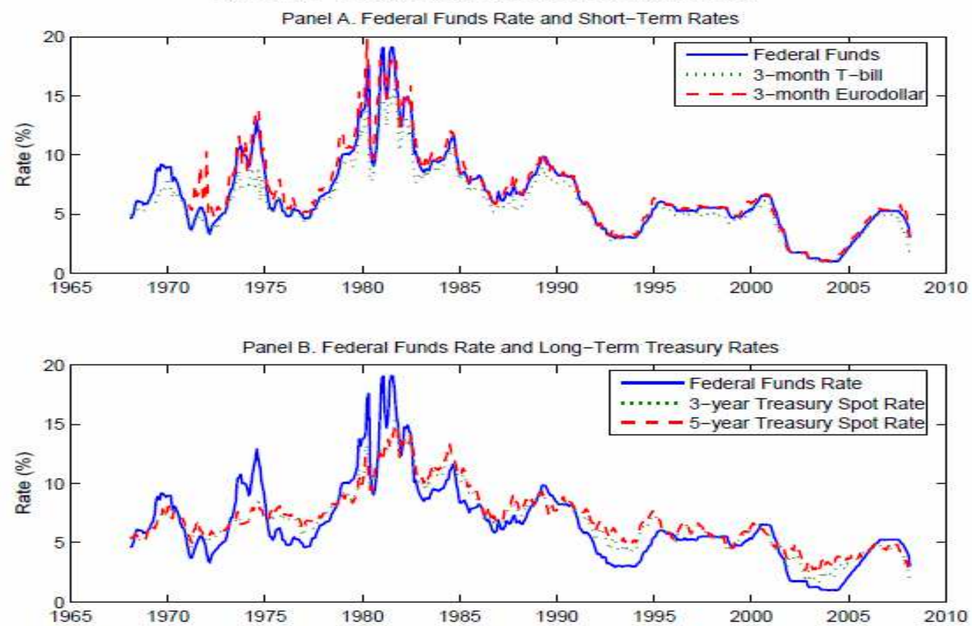
- The **Federal Funds Target (FFT) Rate** is determined by a meeting of the members of the **Federal Open Market Committee (FOMC)** which normally occurs eight times a year about seven weeks apart. The FOMC may also hold additional meetings and implement target rate changes outside of its normal schedule.
  - The Fed uses **open market operations (OMO)** (buy/sell government bonds on the open market) to influence the supply of money in order to drive the EFF rate close to the FFT rate.
  - The FFT rate is also known as the **neutral (or nominal) federal fund rate**.
  - Since December 16, 2008, the FOMC has fixed a target range $0\% - 0.25\%$.


Federal Funds Target Rate (DISCONTINUED SERIES) (DFEDTAR)
Source: Board of Governors of the Federal Reserve System
(Percent)
12.5
10.0
7.5
5.0
2.5
0.0
1980
1985
1990
1995
2000
2005
2010
Shaded areas indicate US recessions.
2012 research.stlouisfed.org
FRED


Federal Funds Target Range - Upper Limit (DFEDTARU)
Federal Funds Target Range - Lower Limit (DFEDTARL)
Federal Funds Target Rate (DISCONTINUED SERIES) (DFEDTAR)
Effective Federal Funds Rate (DFF)
(Percent)
6
5
4
3
2
1
0
-1
2007
2008
2009
2010
2011
2012
2013
DFEDTARU
DFEDTARL
DFEDTAR
DFF
Shaded areas indicate US recessions.
2012 research.stlouisfed.org
FRED

From Veronesi (2010, Chapter 7)

Figure 7.1 Federal Funds Rate and other Interest Rates



Source: Federal Reserve.

☐ Contrary to the Fed (or the Bank of England and the Swiss National Bank), the ECB does not have an explicit interest-rate target.

- However, its aim is explicitly to *"influence money market conditions and steer short-term interest rates*" (ECB, 2011).

- In order to influence short-term money market rates, a **shortage of liquidity** is determined by imposing reserves to Euro area banks.

  $\rightarrow$ they are required to hold compulsory cash deposits on accounts with the Eurosystem, and these requirements are based on the amount and profile of liabilities standing out from the banks balance sheet at the end of each month.

- At the same time, banks in the Eurosystem can refinance themselves through the ECBs weekly Main Refinancing Operations (MROs).

- In these refinancing operations, the ECB returns (provides) liquidity to the market, by allowing banks to tender for cash, against collateral.

- The Euro area equivalent of the FFT rate is European Central Bank's weekly **Main Refinancing Operation (MRO) rate**. Changes in the policy rate are decided during the first of the bimonthly meetings of the ECBs Governing Council.

☐ Since October 2008, in a financial crisis framework, the Eurosystem adopted a **fixed-rate full allotment (FRFA)** tender procedure: in other words, the ECB accommodates any demand for liquidity of bank counterparties at the MRO rate (against eligible collateral) in unlimited amounts.

- In the Eurosystem we observe two additional policy rates forming a symmetric corridor around the main policy (MRO) rate:
  - the lower bound of the corridor, named **deposit-facility rate**, is the rate at which counter-parties can deposit funds overnight in the Eurosystem.
  - the upper bound of the corridor, named **lending-facility rate**, is the rate at which counter-parties can borrow funds overnight from the Eurosystem.

1.2.4 LIBOR, EURIBOR and EONIA

☐ LIBOR stands for London Interbank Offered Rate.

- **Definition:** *The rate at which an individual Contributor Panel bank could borrow funds, were it to do so by asking for and then accepting inter-bank offers in reasonable market size, just prior to 11:00 London time.*

- In other words, the bank provide (submit) the rate at which it could borrow unsecured cash from another bank.

- This rate thus represents the bank's perception of its cost of funds in the inter-bank market.

- **Technical features:** LIBOR is calculated and published by Thmson and Reuters on behalf of the *British Banker's Association* (BBA). Each day, the BBA surveys a Panel of banks asking the question implicit in the above mentioned definition.

- Given the submitted rates, BBA eliminate the highest and the lowest 25% and average the remaining 50%. This average rate is the published (at around 11:45 London time) Libor rate.

- This rate is calculated for 10 different currencies (U.S. and Canadian dollar, British Pound, Euro, Japanese Yen, Swiss Franc, ...) across 15 maturities from the over-night to 1 year.

- The number of banks in the Panel is different depending on the concerned currency: we have 18 banks for the US Dollar Libor, 16 banks for the British Pound and 15 for the Euro.

- The update of the banks in the Panel is made twice a year by BBA with the Foreign Exchange and Money Markets Committee (FXMMC). Contributor banks are selected for currency panels in line with three principles:

  1) scale of market activity,

  2) credit rating and

  3) perceived expertise in the currency concerned.

- In the case of the USD Libor, among the contributor banks we have Bank of America, BNP Paribas, Citibank, Credit Agricole, Credit Suisse and HSBC.

- The Libor is widely used as a reference rate for many financial instruments (interest rates derivatives): Forward rate agreements (FRA), Interest rate futures (e.g., Eurodollar futures), Interest rate swaps/options/cap/floor.

☐ EURIBOR stands for Euro Interbank Offered Rate and is produced by the European Banking Federation (EBF) since 1999.

- **Definition:** *It is the rate at which Euro interbank term deposits within the Euro zone are offered by one Prime Bank to another Prime Bank. It is computed as an average of daily quotes provided for fifteen maturities by a panel of 43 of the most active Banks in the Euro zone. It is quoted on an act/360 day count convention, and is fixed at 11:00am [CET] displayed to three decimal places..*

- Banks in the EURIBOR panel are thus asked to submit rates that reflect the "best" lending rates in unsecured cash transactions that *could* take place in the euro area between the "best banks".

- The rates submitted by the panel banks are therefore, by definition, independent of the situation of the banks submitting those rates or of actual transactions in which they engage.

- EURIBOR is calculated as a trimmed average since the average of the contributions is calculated after eliminating the 15% highest and lowest contributions. The daily calculation is performed by Thomson Reuters on behalf of the EBF.

  - The banks that belong to the panel have been selected to ensure that the diversity of the euro money market is adequately reflected and are banks which are "top-rated by international rating agencies".

  - Banks provide daily quotes in euros for 15 maturities from 1 week to 1 year.

- Panel banks are the banks with the highest volume of business in the euro area money markets and it is made up of:

  1) banks from EU countries participating in the euro from the outset;

  2) banks from EU countries not participating in the euro from the outset and;

  3) large international banks from non-EU countries but with important euro zone operations.

- Among these banks, we have: BNP-Paribas (FR), HSBC France (FR), Société Générale (FR), Deutsche Bank (DE), Commerzbank (DE), Barclays Capital (UK), Citibank, J.P. Morgan Chase Co, Bank of Tokyo Mitsubishi.

☐ Since April 2012, the EBF has also started to produce a USD EURIBOR reference which consists of contributions from a panel of 20 banks consisting of banks from both EU countries and some large international banks from non-EU countries. According to the EBF, USD EURIBOR is...

- **Definition**: *... the rate at which USD interbank term deposits are being offered by one panel bank to another panel bank at 11.00 a.m. Brussels time. All maturities, other than overnight, are quoted for spot value and on an actual / 360 day basis.*

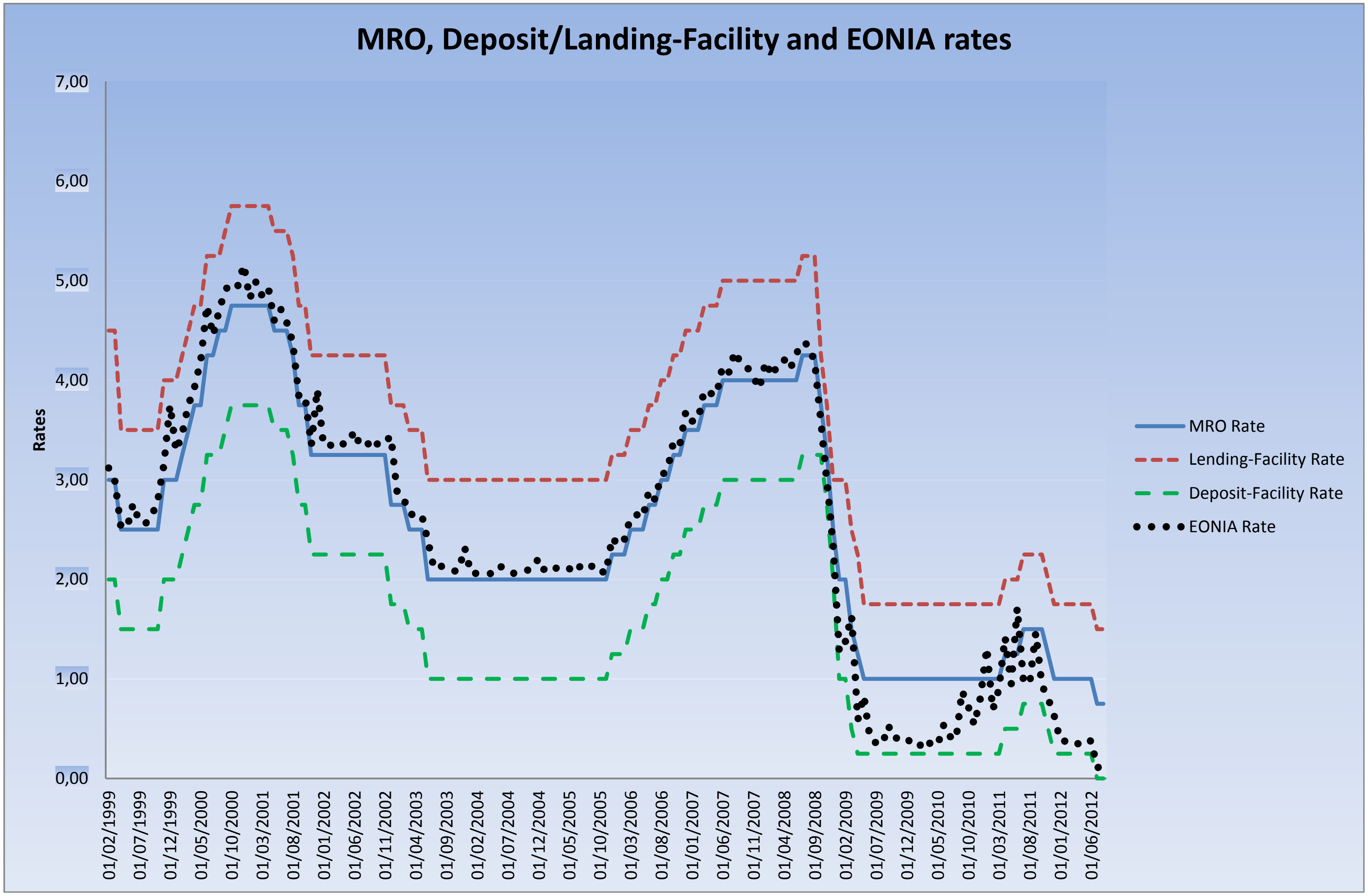
- The choice of banks quoting for USD Euribor is based on market criteria. These international and European banks are of first class market standing and they have been selected to ensure that the diversity of the European US dollar money market is adequately reflected.

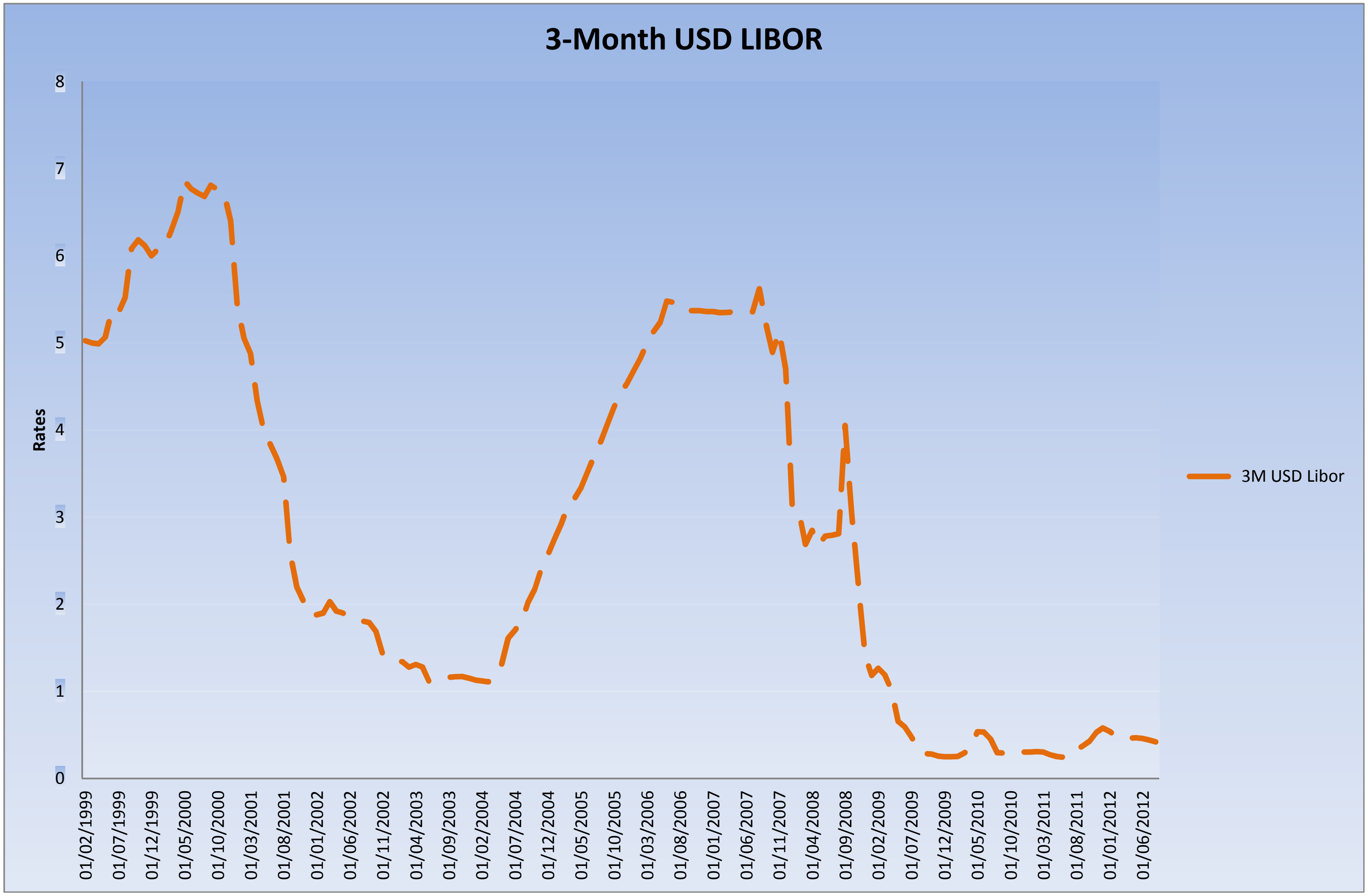
- The definition of USD EURIBOR is therefore also different from that of EUR EURIBOR as the first explicitly links the rates being submitted by the panel banks to their own situation, whereas the second does not. USD EURIBOR, however, is not yet widely used or known in the market.

☐ EONIA stands for Euro Overnight Index Average. According to the EBFs:

- **Definition:** *It is an an effective overnight rate computed as a weighted average of all overnight unsecured lending transactions in the interbank market, initiated within the euro area by the contributing panel banks. Daily reports are provided by the same panel of 43 banks that contribute to the EURIBOR panel. EONIA is calculated by the ECB between 6:45 pm and 7:00 pm (CET) and displayed to three decimal places.*

- It is one of the two benchmarks for the euro area money market with the Euribor.


MRO, Deposit/Landing-Facility and EONIA rates
Rates
7,00
6,00
5,00
4,00
3,00
2,00
1,00
0,00
01/02/1999
01/07/1999
01/12/1999
01/05/2000
01/10/2000
01/03/2001
01/08/2001
01/01/2002
01/06/2002
01/11/2002
01/04/2003
01/09/2003
01/02/2004
01/07/2004
01/12/2004
01/05/2005
01/10/2005
01/03/2006
01/08/2006
01/01/2007
01/06/2007
01/11/2007
01/04/2008
01/09/2008
01/02/2009
01/07/2009
01/12/2009
01/05/2010
01/10/2010
01/03/2011
01/08/2011
01/01/2012
01/06/2012
MRO Rate
Lending-Facility Rate
Deposit-Facility Rate
EONIA Rate


3-Month USD LIBOR
Rates
8
7
6
5
4
3
2
1
0
01/02/1999
01/07/1999
01/12/1999
01/05/2000
01/10/2000
01/03/2001
01/08/2001
01/01/2002
01/06/2002
01/11/2002
01/04/2003
01/09/2003
01/02/2004
01/07/2004
01/12/2004
01/05/2005
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01/03/2006
01/08/2006
01/01/2007
01/06/2007
01/11/2007
01/04/2008
01/09/2008
01/02/2009
01/07/2009
01/12/2009
01/05/2010
01/10/2010
01/03/2011
01/08/2011
01/01/2012
01/06/2012
3M USD Libor

☐ We observe that the EONIA rate tends to track quite closely the MRO rate. Nevertheless, we observe an exception during 2009: the former is well below the latter. Why ?

→ Once the ECB announced the FRFA procedure, an excess of liquidity supply affected overnight trades thus reducing the EONIA towards the lower limit of the monetary policy corridor.

☐ The Swiss National Bank (SNB) sets a target range for the reference interest rate: it is a corridor around (1 percentage point apart) the 3-month Libor rate CHF.

1.2.5 The REPO Market

☐ A **Repurchase Agreement (Repo)** is an agreement to sell (today) some securities to another party and buy them back at a fixed future date and for a fixed amount:

$\rightarrow$ the price at which the security is bought back is greater than the selling price and the difference implies an interest called Repo Rate;

$\rightarrow$ it can be seen as a collateralized loan. The advantage of this borrowing of funds is that the applied (Repo) rate is less than the cost of banking financing.

☐ A **Reverse Repo** is the opposite transaction, namely, it is the purchase of the security for cash with the agreement to sell it back to the original owner at a predetermined price, determined, once again, by the Repo Rate.

$\rightarrow$ it is the same repurchase agreement from the repo dealer's viewpoint, not the trader's.

☐ **Repo Scheme:** A trader *would like* to take at date t a long position (buy!) until date $T > t$ on a given amount of 30-year Treasury bonds.

$\hookrightarrow$ Where does the trader obtains *the funds to finance (a large part of) this position*?

Let P_t denotes the price of that bond at date t.

Date t (Opening Leg). The trader enters a repurchase agreement with a repo dealer:

- the repo dealer gives to the trader the needed cash to buy the bond
- the trader delivers the bond (just bought) as a collateral to the repo dealer;
- in fact, the repo dealer typically gives something less than the market price P_t of the bond, namely $P_t - haircut$, where the $haircut$ implicitly discount the riskiness of the trader (default);
- the trader and the repo dealer agree that the trader will return back the amount borrowed, $(P_t - haircut)$, plus the **repo rate**.

T = t+n (Closing Leg). The trader gets back the bond from the repo dealer:

- the trader sells the bond in the market to get P_T and pays $P_t - haircut$ plus the **repo interest** to the repo dealer, where:

$$\textit{repo interest} = \frac{n}{360} \times \textit{repo rate} \times (P_t - haircut),$$

where $n = T - t$ and 360 is the day count convention in the repo market.

- The profit to the trader is then $P_T - P_t -$ *repo interest* while:

$$\textit{return on capital for trader} = \frac{P_T - P_t - \textit{repo interest}}{haircut}$$

being only the $haircut$ (the margin) the trader's own capital used in the repo.

- It is important to remember that:

  - the term T of the repo transaction is decided at t; most of the agreements are for a very short term, mainly overnight (i.e., one day; it is called *overnight repo*). However, we have also longer-term agreements reaching 30 days or even more (called *term repo*);

  - the repo rate is decided at t;

- The trade implies:

  - a *positive carry* if the return on the bond is *above* the repo rate;

  - a *negative carry* if the return on the bond is *below* the repo rate.

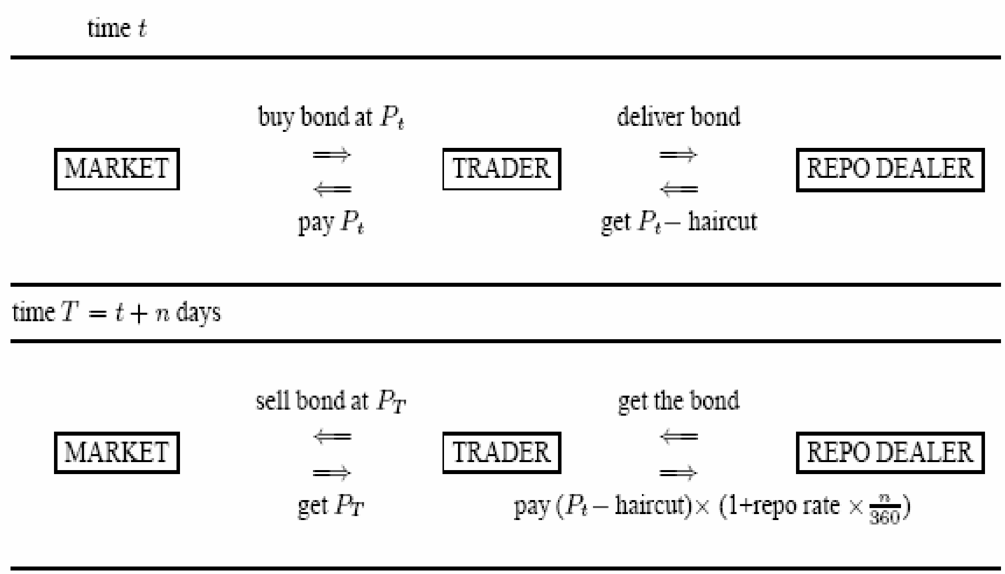
☐ It is also important to highlight the following point: legal title to the bond passes from the trader to the repo dealer.

- Nevertheless, coupons provided by the bond, when the repo dealer owns that asset, are usually passed directly to the trader.

- The agreement might anyway provide that the repo dealer receives the coupons, with the cash payable on repurchase being adjusted to compensate.

From Veronesi (2010, Chapter 1)

Figure 1.4 Schematic Repo Transaction



☐ **Example 1:** At a given date t Institution Y (repo dealer) delivers to Institution X (trader) an amount of (needed) 1,000,000 dollars required to buy a Treasury note with a market value of $P_t = 1,000,000$ dollars (haircut = 0).

- At the same time Institution X gives the bond to Institution Y as collateral.

- On the sale day (date t) Institution X agrees to buy back from Institution Y the same security on the next day (overnight) at a price $P_T = 1,000,138.89$

- The price P_T has been determined through a repo rate of 5% that the lender of cash (Institution Y) apply to the borrower (Institution X):

  $1,000,000 \times \left(1 + 0.05\,\frac{1}{360}\right) = 1,000,138.89.$

☐ Other important definitions and characteristics of the repo market are as follows:

1. **General Collateral Rate (GCR):** This is the repo rate on most Treasury securities, such as the off-the-run (i.e., *not the most recently issued*) Treasuries. Because most Treasury securities have similar characteristics in terms of liquidity, market participants require the same interest rate for collateralized borrowing.

2. **Special Repo Rate:** At times, one particular Treasury security is in high demand and hence the repo rate on that security falls to a level below the GCR. As an example, on-the-run ((i.e., *the most recently issued*)) Treasury securities typically are " *on special* ", in the sense that the repo rate is smaller than the GCR.

☐ For which reason a security which is high in demand entail a lower (special) repo rate ? To understand the logic, consider the next example which considers the *reverse repo*.

- **Reverse Repo Scheme:** Let us consider an investor who thinks that a particular bond, such as the on-the-run 30-year treasury bond, is overpriced and wants to take a bet that its price will decline in the future.

  – If the trader does not have the bond to sell outright, then he/she can enter into a reverse repo with a repo dealer to obtain the bond to sell.

- More specifically, in a reverse repo, the trader at date t (Opening Leg):

(A) borrows the security (the bond) from the dealer;

(B) sells it in the market at the price P_t;

(C) use P_t as cash collateral for the repo dealer

- Here, the trader is *lending* money to the repo dealer, against the bond, and thus he is entitled to receive the repo rate.

- At date $T = t + n$ (Closing Leg), the trader:
  - buys the bond in the market paying the price P_T;
  - gives the bond back to the repo dealer;
  - obtains from the repo dealer $P_t \times (1 + \textit{repo rate} \times \frac{n}{360})$.
- The profit of the trader is $(P_t - P_T) + \textit{repo interest}$, where:

$$\textit{repo interest} = \frac{n}{360} \times \textit{repo rate} \times P_t \,.$$

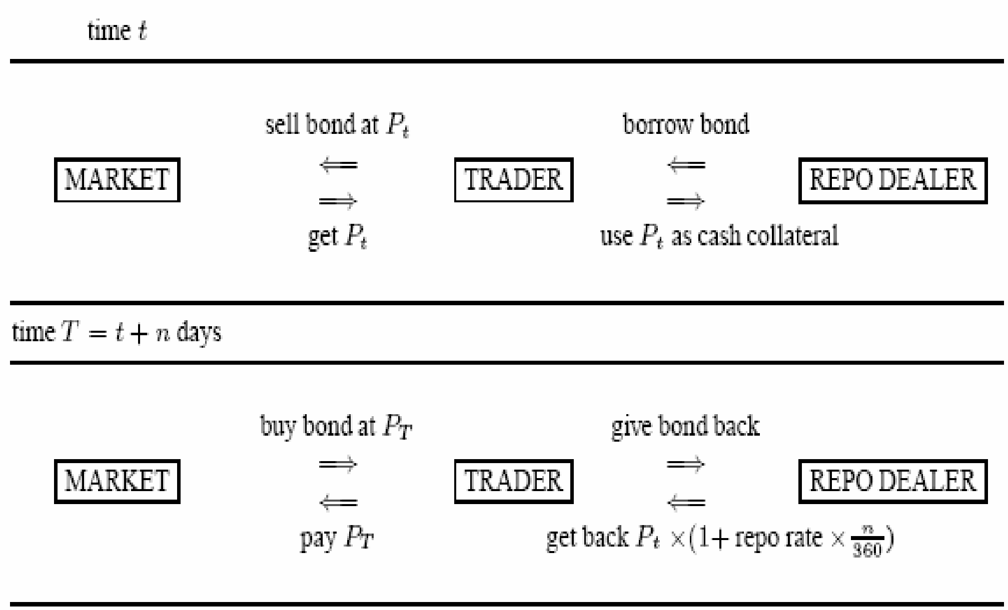
☐ **Example 2:** At a given date t Institution X borrows a Treasury note with a market value of $P_t = 1,000,000$ dollars to Institution Y and delivers to Institution Y 1,000,000 dollars of cash as collateral.

- On the sale day (date t) Institution X agrees to sell back to Institution Y the same security on the next day (overnight) at a price $P_T = 1,000,138.89$, determined through a repo rate of 5% that the lender of cash (Institution X) apply to the borrower of cash (Institution Y)

- Institution X is said to have a reverse repo position. It has effectively a short position in the Treasury note. Indeed, the profitability of Institution X (bond trader) goes up when the Treasury note price goes down ($P_t > P_T$).

From Veronesi (2010, Chapter 1)

Figure 1.5 Reverse Repo Transaction



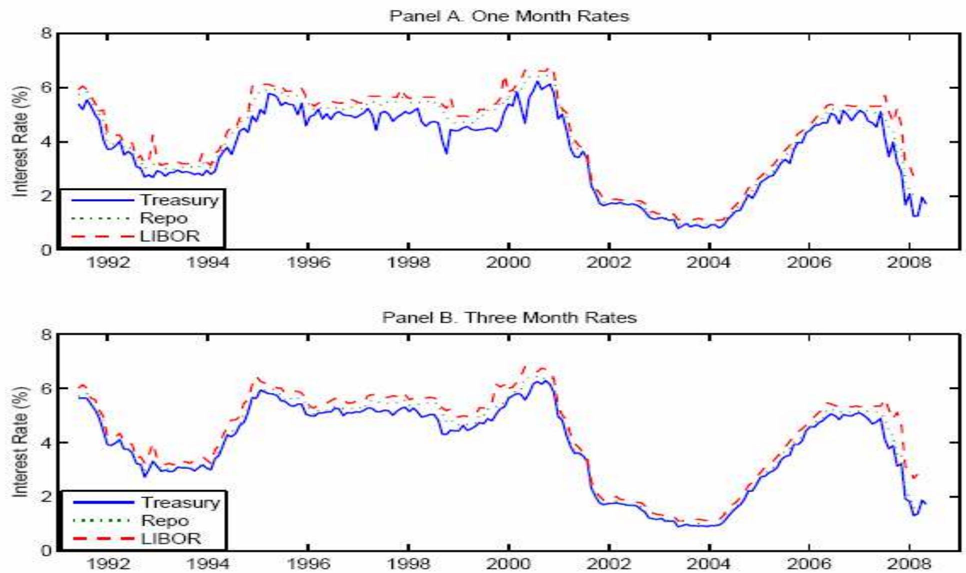
- Now, the trader who wants to speculate on the decrease in the bond price, is happy to accept a reduction of the repo rate in order to get hold of the bond and realize his/her trading strategy.

- If many traders want to undertake the same strategy of shorting that particular bond, then the bond is high in demand (to repo dealers) and, thus, the repo rate for *that* bond declines below the general collateral rate. The bond is said to be *on special*.

- Since borrowing is collateralized by the value of the asset, the repo rate is lower than other borrowing rates available to banks, such as the (more risky, being non collateralized) Libor. It is higher than (risk-less) Treasury rates.

From Veronesi (2010, Chapter 1)

Figure 1.6 Short-Term Rates: 1991 - 2008



Source: Federal Reserve Board, British Bankers Association, Bloomberg.

1.3 BASICS OF FIXED INCOME SECURITIES

1.3.1 Time value of money

□ **Periodically Compounded Interest Rate :** is the concept of adding accumulated interest back to the principal.

Example : an amount of 1500 dollars (the principal) is deposited at date t in a bank paying an annual interest rate of 4.3%, compounded quarterly. Which is the value of the principal after 6 years ?

$$
\begin{aligned}
A_1 &= 1500 \times (1 + 0.043/4)\,, \text{ value of the principal after 1 quarter} \\
A_2 &= A_1 \times (1 + 0.043/4) = 1500 \times (1 + 0.043/4)^2\,, \text{ ... after 2 quarters} \\
&\vdots \\
A &= 1500 \times (1 + 0.043/4)^{4\times 6} = 1938.8\,, \text{ value of the principal after 6 years}\,.
\end{aligned}
$$

Remark : 4.3% is the **nominal rate** (unadjusted for compounding!), while the equivalent **effective annual rate** is 4.37% (adjusted for periodic compounding):

$$1500 \times (1 + 0.0437)^6 = 1938.8\,.$$

☐ More formally - The value A of the principal P after n years at the nominal annual rate R compounded m times per year is:

$$A = P \times (1 + R/m)^{mn}\,.$$

☐ The associated **effective annual rate** is the rate R^{eff} such that:

$$(1 + R^{eff}) = (1 + R/m)^m \;\Rightarrow\; R^{eff} = (1 + R/m)^m - 1\,.$$

☐ **Present and Future Values :** Let us consider a nominal interest rate R per year, compounded m times per year. The **present value** $V_P(m)$ of 1 unit of money to be received in n years is:

$$V_P(m,n) = (1 + R/m)^{-mn}\,, \text{ over } n \text{ years}$$

The **future value** of 1 unit of money capitalized m times per year at the rate R for n years is:

$$V_F(m,n) = (1 + R/m)^{mn}\,, \text{ over } n \text{ year}$$

☐ $V_F(m,n) = 1/V_P(m,n)$.

☐ **Continuously Compounded Interest Rate :** if we consider a compounding period infinitely small, that is if $m \to \infty$, we have:

$$\begin{aligned} V_P(\infty, n) &= \lim_{m\to\infty} V_P(m,n) = \exp(-R_\infty n)\,, \text{ over } n \text{ years} \\ V_F(\infty, n) &= \lim_{m\to\infty} V_F(m,n) = \exp(R_\infty n)\,, \end{aligned}$$

and R_∞ denotes the **nominal annual continuously compounded rate**. It is also called **force of interest**. The associated **effective annual rate** is:

$$R_\infty^{eff} = \exp(R_\infty) - 1\,.$$

□ Comparing $V_P(\infty, 1)$ and $V_P(m, 1)$ we find the following relation between the annual continuously compounded and periodically compounded rates:

$$(1 + R/m)^m = \exp(R_\infty) \quad \Rightarrow \qquad R_\infty = m \ln(1 + R/m)\,,$$
$$R = m \exp(R_\infty/m) - m\,.$$

Example : Let us consider an annual rate $R = 15\%$ compounded quarterly ($m = 4$). The equivalent continuously compounded rate R_∞ is : $R_\infty = 4 \ln(1 + 0.15/4) = 0.147$.

- **Simple Interest Rate (linear capitalization):** in contrast to periodically compounded interest rates, when we discount or capitalize using simple interest rates, the interest generated at each period is not added to the principal. We also refer to *simply compounded* rates.

- The future value of 1 unit of money invested for n years at the annual rate R is:

$$V_F^n = (1 + R \times n)\,, \text{ over } n \text{ years, and } V_F^n = 1/V_P^n.$$

Example : Given at date t a principal of $P = 2500$ dollars and a simple annual interest rate $R = 12.99\%$, its future value after 3 months is:

$$V_F^{0.25} = 2500 \times (1 + 0.1299 \times 0.25) = 2580.6\,.$$

□ Let us consider $m = 1$ (and $R > 0$), then $V_F(1, n) = (1 + R)^n \geq V_F^n = (1 + R \times n)$. The equality is attained when $n = 1$.

Example : Let us take $P = 2500$, $n = 5$, $m = 1$ and an annual rate $R = 0.03$. Depending on the kind of compounding (simply or periodically) we have $V_F^5 = 2875$ and $V_F(1, 5) = 2898.2$. If $m = 4$, we have $V_F(4, 5) = 2903$ and, if $m = \infty$, $V_F(\infty, 5) = 2905$.

1.3.2 Discount Factors: Bonds

- A **zero-coupon bond** (ZCB) is the simplest possible bond : it promises a **single and predetermined** payment at the maturity date. This payment is called **face value**, **par value** or **principal**. It is the amount of money paid to the bond holder at the maturity date.

- A **coupon bond** is a bonds which promises, not only the face value at the maturity date, but also other regular payments (coupons) between the date of issue and the maturity date (included).

- The face value will be assumed (in general) equal to 1 unit of money (dollar).

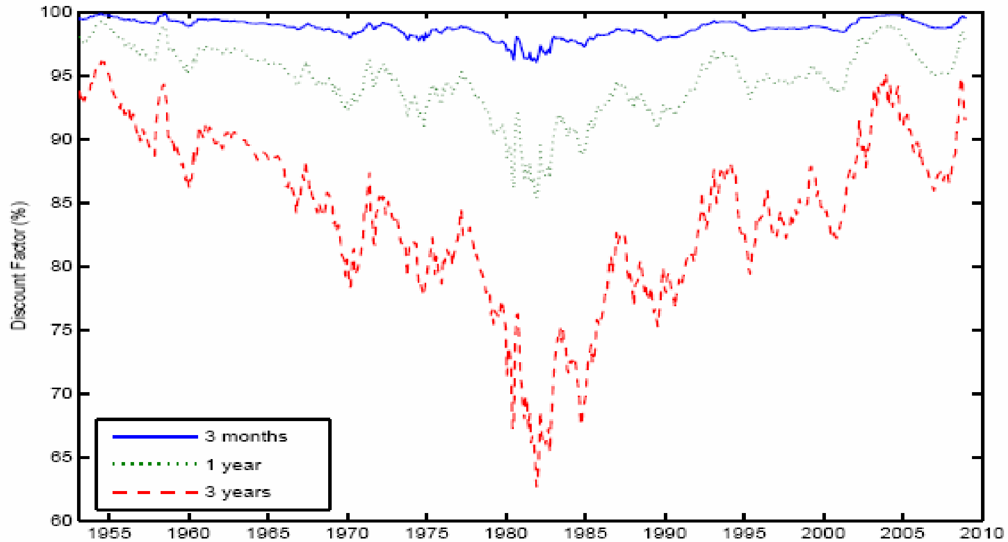
- Let us denote by $B(t,T)$ the market value at the date t (date of potential issue) of a zero-coupon bond, with unitary face value, maturing at date T (the maturity date).

- The market price $B(t,T)$ represent the value at t for a "sure" payment at T of 1 unit of money $\Rightarrow B(t,T)$ reflects the **market Discount Factor** of sure date T payments.

- Indeed, if the ZCB face value at T is C_T (say), then its price at t is $C_T B(t,T)$: the sure payment at T is discounted (actualized) at t by the Discount Factor $B(t,T)$.

From Veronesi (2010, Chapter 2)

Figure 2.1 Discount Factors



Source: Center for Research in Security Prices (CRSP)

- If at the date t, several ZCBs with different maturity dates are traded, we can form the function $T \to B(t,T)$, called **market Discount Function** prevailing at t.

- It is obvious that $B(t,t) = 1$ and $B(t,S) < B(t,T)$, $\forall S > T$. More precisely, the discount function should be decreasing in the maturity date :

  $0 < B(t,S) < B(t,T) < 1$, $S > T$.

- A coupon bond has multiple payment dates T_i, $\{1,\ldots,n\}$, $T_1 < \ldots < T_n$ (with $T_n = T$ the maturity date). The payment (coupon) at date T_i is given (in general) by the coupon rate c_i times the face value C_T. At T, the holder receives also C_T.

- In general, payments occur at regular intervals (6 months for T-bills, T-bonds) so that, for all i, $T_{i+1} - T_i = \Delta_i = \Delta$ for some fixed time interval Δ. If we measure time in "years", typical bonds have $\Delta \in \{0.25, 0.5, 1\}$ corresponding to quarterly, semi-annual or annual payments.

- In many cases coupon rates c_i are quoted as an annual rate even if they are paid more frequently $\Rightarrow$ the periodic coupon rate is $c_i\Delta$.

- Let us consider the (classical) so-called *bullet bonds* or *straight-coupon bonds* : here, all payments before the maturity date (T_i, $i < n$) are given by the product of the coupon rate (c_i) and the face value (C_T). The payment at the maturity date $T_n = T$ is the sum of the coupon plus the face value.

- More formally, given an annual coupon rate of c_i, the payments C_i (say) at the dates $T_1 < \ldots < T_n$, per unit of face value ($C_T = 1$), are:

$$C_i = \begin{cases} c_i\Delta\,, \ \forall i \in \{1, \ldots, n-1\} \\ c_i\Delta + 1\,, \ i = n\,. \end{cases}$$

For $c_i = 0$, $\forall i \in \{1, \ldots, n\}$, we are back to a ZCB.

- A coupon bond can be seen as a portfolio of ZCBs : C_1 units of ZCBs maturing at T_1, C_2 units of ZCBs maturing at T_2,, etc. If all these ZCBs are traded in the market, the price at t of the coupon bond must be:

$$CB(t,T) = \sum_{i=1}^{n} 1_{\{t<T_i\}} C_i B(t,T_i)\,, \ \text{otherwise A.O.!} \tag{1}$$

- **Example :** Let us consider at date t, a bullet bond with $C_T = 100$, an annual coupon rate of 7% and time-to-maturity 3 years ($T - t = 3$ years). Suppose that, at date t, ZCBs with face value equal to 1 dollar, are traded with residual maturity of 1, 2 and 3 years. Assume that the market prices are $B(t, t + 1y) = 0.98$, $B(t, t + 2y) = 0.94$ and $B(t, t + 3y) = 0.90$. The price of the bullet bond must be:

$$
\begin{aligned}
CB(t, t + 3y) &= (0.07 \cdot 100) \cdot 0.98 + (0.07 \cdot 100) \cdot 0.94 \\
&\quad + [(0.07 \cdot 100) + 100] \cdot 0.90 = 109.74.
\end{aligned}
$$

$\hookrightarrow$ if $CB(t, t+3y) < 109.74 \Rightarrow$ the market provide an **arbitrage opportunity**, that is a **risk-free profit** : at date t sell 7 one-year, 7 two-year and 107 three-year ZCBs (earning 109.74), and buy the bullet bond at the market price $CB(t, t + 3y)$.

- You obtain (for sure!) a positive net profit at t of $109.74 - CB(t, t+3y)$.

$\hookrightarrow$ if $CB(t, t+3y) > 109.74 \Rightarrow$ you have again an arbitrage opportunity! Why ?

- If at date t the market does not trade all relevant ZCBs that we need to replicate the coupon bond, we cannot justify relation (1) via the replication argument. Nevertheless, the relation (based on the no-arbitrage principle) is still correct! If an investor has determined a discount function $B(t, T_i)$ (using private/macroeconomic information), then he can us it to implement (1).

- The market prices of all ZCBs reflect a market Discount Function which is a complex average of individual discount functions of market participants. This market Discount Function is clearly the result of supply and demand of ZCBs from all market participants.

- In most markets only a few ZCBs are traded, so that information about the discount function must be inferred from market prices of coupon bonds. We will discuss that point in **Lecture 2**.

  Remark : In general, coupon bonds have a fixed coupon rate ($c_i = c$), but a small minority of bonds have coupon rates that are reset periodically over the life of the bond (between t and T) $\rightarrow$ *floating rate bonds*.

1.3.3 Bond market conventions : a) Day-count conventions

- The market generally measures time in years. If t and T denotes two dates expressed as day/month/year, it is not clear what the time-to-maturity (or, residual maturity) $T-t$ should be.

- The market evaluates the year fraction between t and T in different ways :

□ $T-t = Actual/365$: 1 year = 365 days, and the day-count convention gives:

$$T-t = \frac{\text{actual number of days between } t \text{ and } T}{365}.$$

□ $T-t = Actual/360$ (money market/T-bills): as above but the year counts 360 days.

☐ $T - t = Actual/Actual$ (T-notes and T-bonds):

$$T - t = \frac{\text{actual number of days between } t \text{ and } T}{\text{actual number of days in the year}}.$$

☐ 30/360 (US corporate and municipal bonds): months count 30 and years 360 days. Let $t = d_1/m_1/y_1$ and $T = d_2/m_2/y_2$. Then, this day-count convention gives:

$$T - t = \frac{\min(d_2, 30) + (30 - d_1)^+}{360} + \frac{(m_2 - m_1 - 1)^+}{12} + y_2 - y_1 .$$

☐ **Example :** the time between t = 04/01/2000 and T = 04/07/2004 is in the 30/360 convention:

$$\frac{4 + (30 - 4)}{360} + \frac{(7 - 1 - 1)}{12} + 2002 - 2000 = 2.5\,.$$

b) Accrued Interest, Clean Price and Dirty Price

Let us first make a couple of remarks about the coupon bond price :

- The definition of coupon bond price (1) provides an ex-dividend price. The cum-dividend price is obtained by replacing the strict inequalities by inequalities.

- Given an initial date $t < T_1$, and a coupon bond price $CB(t,T)$, as far as $t \to T$ the number of terms in (1) decreases:

$$CB(t,T) = \sum_{i=1}^{n} C_i B(t,T_i)\,,\ t \leq T_1\,,\ \searrow\ CB(t,T) = \sum_{i=2}^{n} C_i B(t,T_i) \text{ when } t \in (T_1,T_2]\,.$$

$\to$ more precisely, the price function $t \mapsto CB(t,T)$ has systematic discontinuities ("jumps") at dates $t = T_1, T_2, \ldots, T_n$ because of coupon payments.

- This is why the bond market distinguishes between *clean price* (or, quoted price) and *dirty price* (or, cash price).

- Let us first introduce the notion of *accrued interest* at time $t \in (T_{i-1}, T_i]$:

$$AI(i;t) = C_i \times \frac{t - T_{i-1}}{T_i - T_{i-1}},$$

where date differences are taken according to a certain day-account convention.

- The *clean price* at date t of the coupon bond is

$$CB^{clean}(t,T) = CB(t,T) - AI(i;t)\,, \quad t \in (T_{i-1}, T_i]\,.$$

- In other words, when we buy a coupon bond quoted at the clean price $CB^{clean}(t,T)$ at date $t \in (T_{i-1}, T_i]$, the cash price we have to pay is:

$$CB(t,T) = CB^{clean}(t,T) + AI(i;t)\,, \quad t \in (T_{i-1}, T_i]\,.$$

1.3.4 Bond yields and zero-coupon rates

- Even if discount factors ($B(t,T)$) provide full information about how to discount future sure payments, it is much easier to interpret (and compare) the information provided by interest rates (a given percentage per year).

- Nevertheless, to correctly use and assess the magnitude of an interest rate, we need to know the **compounding frequency** of that rate.

- Moreover, it is important to know **at which time the rate is set** or observed, and over which period of time the interest rate **applies**.

- Here, we will introduce **spot rates** (they apply to a period beginning at the time the rate is set), and then we will present **forward rates** (they apply to a period beginning after the date they set).

☐ **Yield of a coupon bond** - Given at date t a coupon bond with maturity date T, price $CB(t,T)$ and payments $\{C_1,\ldots,C_n\}$ at dates $\{T_1,\ldots,T_n\}$, respectively, the associated **annually compounded yield to maturity** is the value $Y^{CB}(t,T)$ satisfying the equation:

$$CB(t,T) = \sum_{i=1}^{n} C_i (1 + Y^{CB}(t,T))^{-(T_i - t)}\,.$$

Note that the same discount rate is applied to all payments!

□ **Yield of a ZCB** with $C_T = 1$ at the maturity date T - The **annually compounded yield to maturity** $(m = 1)$ is the value $Y(t,T)$ satisfying the equation:

$$B(t,T) = (1 + Y(t,T))^{-(T-t)},$$

and therefore we have $Y(t,T) = B(t,T)^{-1/(T-t)} - 1$;

→ we call $Y(t,T)$ the annually compounded **zero-coupon yield**, or **zero-coupon rate**, or **spot rate** for date T.

→ the function $T \mapsto Y(t,T)$ is called the (annually compounded) **zero-coupon yield curve** or simply the **yield curve**. It provides the same information as $T \mapsto B(t,T)$.

☐ The **continuously compounded yield to maturity** of the above mentioned coupon bond is the value $R^{CB}(t,T)$ (also called *gross redemption yield*) satisfying the equation:

$$CB(t,T) = \sum_{i=1}^{n} C_i \, \mathrm{exp}[-R^{CB}(t,T) \times (T_i - t)] \, .$$

☐ The quoted YTM $\bar{R}^{CB}(t,T)$ (say) is the periodically compounded rate with capitalization frequency (m) equal to the coupon frequency:

$$\bar{R}^{CB}(t,T) = m \, \mathrm{exp}[R^{CB}(t,T)/m] - m \, .$$

Example : If the coupons are semiannual ($m = 2$), then

$$\bar{R}^{CB}(t,T) = 2[\mathrm{exp}[R^{CB}(t,T)/2] - 1].$$

☐ The function $T \mapsto R^{CB}(t,T)$ is also a yield curve providing the same information as $T \mapsto CB(t,T)$ or $T \mapsto Y^{CB}(t,T)$.

☐ In the particular case of a ZCB, the **continuously compounded yield to maturity** $R(t,T)$ satisfies:

$$B(t,T) = \exp[-R(t,T) \times (T-t)]\,, \;\Rightarrow R(t,T) = -\frac{1}{T-t} \ln B(t,T)\,. \tag{2}$$

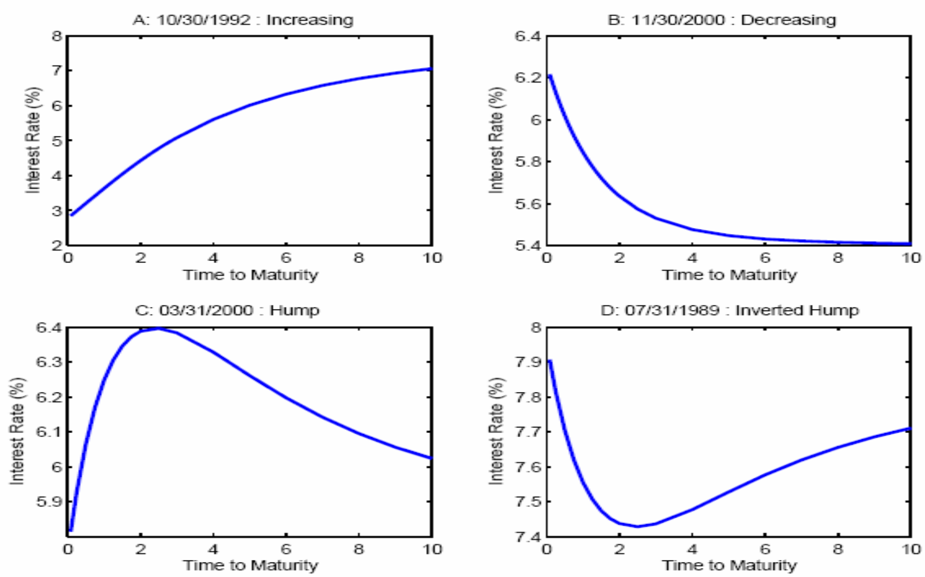
☐ $R(t,T) = \ln(1 + Y(t,T))$, is also called the **term structure of interest rates**.

☐ There is an inverse relation between yield and associated bond price level.

☐ For mathematical convenience we will focus on $R(t,T)$ in most models.

From Veronesi (2010, Chapter 2)

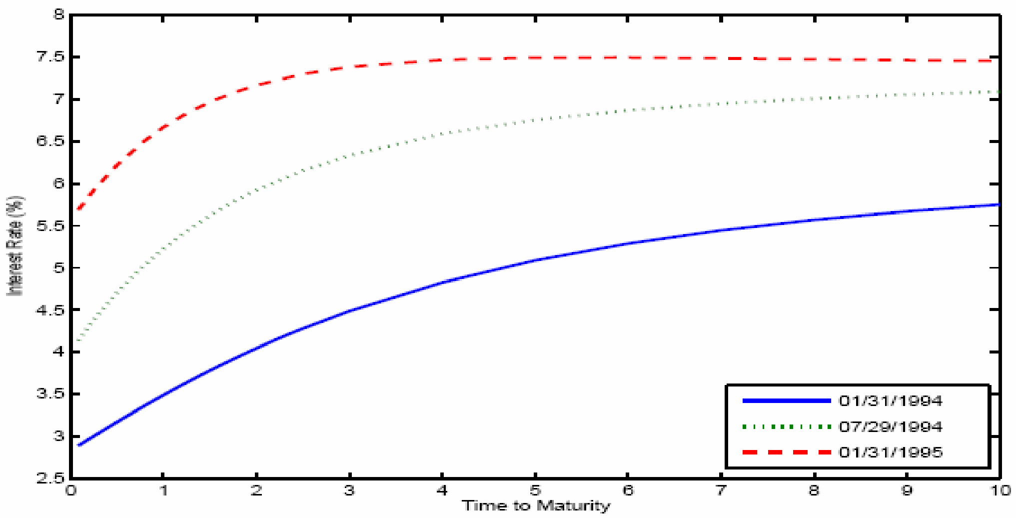
Figure 2.3 The Shapes of the Term Structure



Data Source: Center for Research in Security Prices

From Veronesi (2010, Chapter 2)

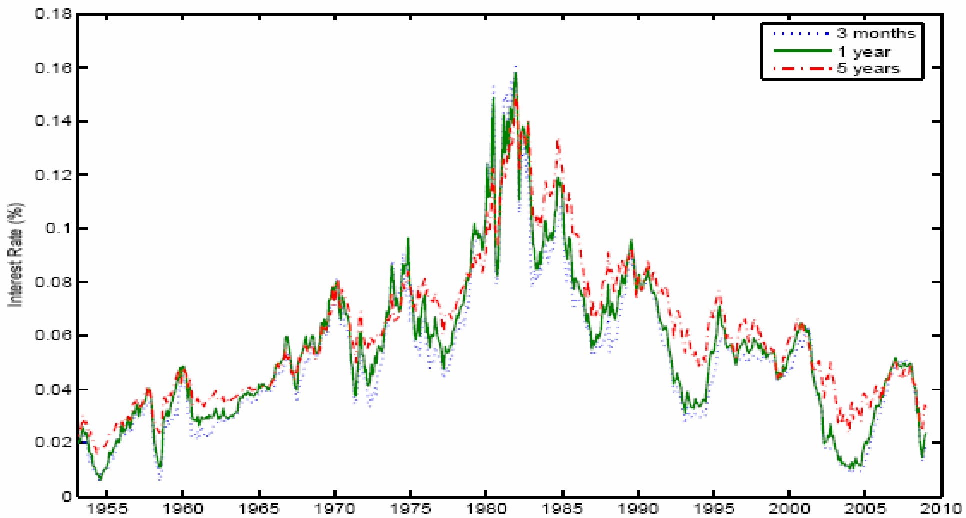
Figure 2.4 The Term Structure of Interest Rates on Three dates



Data Source: Center for Research in Security Prices

From Veronesi (2010, Chapter 2)

Figure 2.5 The Term Structure over Time



Source: Center for Research in Security Prices

Example : Let us assume that at date t we observe on the zero-coupon bond market the following continuously compounded yield curve (annual basis!) :

$T-t$	0.5	1	1.5	2
$R(t,T)$	2.5%	2.0%	2.7%	3.0%

Q : Which is the price at date t of the coupon bond with residual maturity 2 years, $C_T = 100$, semiannual coupons with annual coupon rate of 7% and coupon dates $T_1 = t + 0.5$, $T_2 = t + 1$, $T_3 = t + 1.5$ and $T_4 = t + 2$? Determine its continuously compounded YTM and its quoted YTM.

$A.1$: First, we determine the zero-coupon bond prices :

$$
\begin{aligned}
B(t, t+0.5) &= \exp[-0.025 \times 0.5] = 0.9876 \\
B(t, t+1) &= \exp[-0.02] = 0.9802 \\
B(t, t+1.5) &= \exp[-0.027 \times 1.5] = 0.9603 \\
B(t, t+2) &= \exp[-0.03 \times 2] = 0.9418\,.
\end{aligned}
$$

$\Rightarrow$ the coupon bond price is:

$$
\begin{aligned}
CB(t, t+2) &= 3.5 \times [B(t, t+0.5) + B(t, t+1) + B(t, t+1.5)] \\
&\quad +103.5 \times B(t, t+2) \\
&= 3.5 \times [0.9876 + 0.9802 + 0.9603] + 103.5 \times 0.9418 \\
&= 107.7246\,.
\end{aligned}
$$

$A.2:$ The associated continuously compounded YTM is the rate $R^{CB}(t,t+2)$ such that:

$$
\begin{aligned}
CB(t,t+2) &= 107.7246 = 3.5 \times [e^{-0.5R^{CB}(t,t+2)} + e^{-R^{CB}(t,t+2)} + e^{-1.5R^{CB}(t,t+2)}] \\
&\qquad +103.5 \times e^{-2R^{CB}(t,t+2)}, \\
&\Rightarrow R^{CB}(t,t+2) = 2.97\%.
\end{aligned}
$$

$A.3:$ The quoted YTM, with semiannual coupons, is:

$$\bar{R}^{CB}(t,t+2) = 2(e^{R^{CB}(t,t+2)/2} - 1) = 2.999\%.$$

- The **ZCB holding-period return :** is the return over some holding period less than the bond's residual maturity. Let us assume for ease of exposition that the holding period be a single period.

- Let us define $R_{t,t+1}^{T-t}$ the **arithmetic** one-period holding-period return on a ZCB purchased at time t at the price $B(t,T)$ and sold at time $t+1$ at $B(t+1,T)$. That is:

$$R_{t,t+1}^{T-t} = \frac{B(t+1,T)}{B(t,T)} - 1 \;\Rightarrow\; (1 + R_{t,t+1}^{T-t}) = \frac{(1+Y(t,T))^{T-t}}{(1+Y(t+1,T))^{T-t-1}}$$

- $R_{t,t+1}^{T-t}$ is HIGH if $Y(t,T)$ is HIGH or if $Y(t+1,T)$ is LOW.

☐ Let us take the **geometric** return $r_{t,t+1}^{T-t} = \ln(1 + R_{t,t+1}^{T-t})$, and we find:

$$r_{t,t+1}^{T-t} = \ln\left(\frac{B(t+1,T)}{B(t,T)}\right) = (T-t)R(t,T) - (T-t-1)R(t+1,T)$$
$$= R(t,T) - (T-t-1)(R(t+1,T) - R(t,T))$$

☐ $r_{t,t+1}^{T-t}$ is determined by the beginning-of-period yield $R(t,T)$ (positively) and by the variation of the yield over the holding period (negatively).

☐ We also have that:

$$R(t,T) = \frac{1}{T-t}\sum_{i=0}^{T-t} r_{t+i,t+1+i}^{T-t-i}$$

and the continuously compounded yield on a ZCB equals the average continuously compounded return per period if the bond is held to maturity.

- **Spot LIBOR rate :** the simply compounded interest rate convention is typically applied for commercial loans (fixed and floating reference rate loans). The most commonly used floating reference rate is the LIBOR (London Interbank Offered Rate).

- Let us consider, at date t, the quoted LIBOR annual rate with maturity date T, denoted $L(t,T)$ (say). The present value of 1 unit of money paid at T is:

$$B_L(t,T) = \frac{1}{[1 + L(t,T) \times (T-t)]} \, ,$$

that is:

$$L(t,T) = \frac{1}{T-t}\left(\frac{1}{B_L(t,T)} - 1\right) \, ,$$

☐ If in the market we have at date t a ZCB maturing at T with price $B(t,T)$, for arbitrage reasons we must have $B_L(t,T) = B(t,T)$. Indeed, let us imagine that, at time t, we have $B_L(t,T) > B(t,T)$. In that case, an investor can make a sure positive profit.

$\rightarrow$ He borrows at t, over the period $[t,T]$, the amount of money $B_L(t,T)$ at the simple rate $L(t,T)$ and he buys (again, at t) the ZCB paying the price $B(t,T)$.

$\Rightarrow$ At t he earns $B_L(t,T) - B(t,T)$, while at T he has a null profit : he obtain 1 unit of money from the ZCB and he provide the amount of money $B_L(t,T)[1 + L(t,T) \times (T-t)] = 1$.

1.3.5 Forward rates

- The annually compounded spot rate $Y(t,T)$ set at date t concerns the price on a loan between the same date t (trading and settlement dates coincide) and the maturity date T. In the case of **forward rates**, the loan is received at some future settlement date $\tau \geq t$ and the maturity date is (as usual) $T > \tau \geq t$.

- $t =$ trading date, $\tau =$ settlement date and $T =$ maturity date;

- We are fixing the rate of interest over the period $[\tau, T]$ in advance at time t.

- In other words, this is the rate which is appropriate at time t for discounting between τ and T.

□ The **annually compounded forward rate**, denoted $Y(t,\tau,T)$, with $t \le \tau < T$, is the rate such that:

$$(1+Y(t,T))^{-(T-t)} = (1+Y(t,\tau))^{-(\tau-t)} \times (1+Y(t,\tau,T))^{-(T-\tau)}\,, \qquad (3)$$

(when $t=\tau$ the forward rate reduces to the spot rate) and we deduce that:

$$Y(t,\tau,T) = \frac{(1+Y(t,\tau))^{-(\tau-t)/(T-\tau)}}{(1+Y(t,T))^{-(T-t)/(T-\tau)}} - 1\,.$$

□ we can also write (11) in terms of ZCB prices:

$$B(t,T) = B(t,\tau) \times (1+Y(t,\tau,T))^{-(T-\tau)}\,. \qquad (4)$$

If relation (3) (or (4)) are not satisfied : arbitrage opportunity! Why ?

☐ The **short** (annually compounded) **forward rate** is given by:

$$Y(t,\tau,\tau+1) = \frac{B(t,\tau)}{B(t,\tau+1)} - 1\,.$$

☐ This means that we can define the following relation between the zero-coupon bond price and the sequence of short forward rates applying over time-to-maturity interval:

$$B(t,\tau) = \prod_{j=t}^{\tau-1} \frac{B(t,j+1)}{B(t,j)} = \prod_{j=t}^{\tau-1} \frac{1}{1+Y(t,j,j+1)}\,.$$

□ The **forward LIBOR rate** at date t, valid for the period $[\tau, T]$, is the simply compounded rate $L(t, \tau, T)$ such that:

$$B(t, \tau) = B(t, T) \times [1 + L(t, \tau, T) \times (T - \tau)] \, , \tag{5}$$

and therefore

$$L(t, \tau, T) = \frac{1}{T - \tau} \left(\frac{B(t, \tau)}{B(t, T)} - 1 \right) \, .$$

□ The **continuously compounded forward rate** prevailing at date t, for the period $[\tau, T]$, is the rate $R(t, \tau, T)$ such that:

$$\exp[-R(t, T) \times (T - t)] = \exp[-R(t, \tau) \times (\tau - t)] \times \exp[-R(t, \tau, T) \times (T - \tau)] \, .$$

☐ This means that $R(t, \tau, T)$ is given by:

$$
\begin{aligned}
R(t, \tau, T) &= -\frac{1}{T-\tau} \ln \frac{B(t,T)}{B(t,\tau)} , \\
&= \frac{R(t,T)(T-t) - R(t,\tau)(\tau - t)}{T-\tau} .
\end{aligned}
\tag{6}
$$

☐ the quantity $R(t, \tau, \tau+1) = \ln \frac{B(t,\tau)}{B(t,\tau+1)}$ is called the **short forward rate** and gives the possibility to write:

$$
B(t, \tau) = \exp\left[- \sum_{j=t}^{\tau-1} R(t, j, j+1) \right] . \tag{7}
$$

Example : Let us consider again the yield (and Discount) curve of a previous example:

$\tau - t$	0.5	1	1.5	2
$R(t,\tau)$	2.5%	2.0%	2.7%	3.0%
$B(t,\tau)$	0.9876	0.9802	0.9603	0.9418

$\Rightarrow$ the forward rates $R(t,\tau,\tau+0.5) = -2 \times \ln\left[\frac{B(t,\tau+0.5)}{B(t,\tau)}\right]$ are given by :

$\tau - t$	0	0.5	1	1.5
$R(t,\tau,\tau+0.5)$	2.5%	1.5%	4.1%	3.89%

$\hookrightarrow$ these are annually short forward rates, with short period = 0.5 years!!!

$\Rightarrow$ we can also apply formula (7), using the above determined short forward rates, to find again the zero-coupon bond prices:

$$
\begin{aligned}
B(t, t+0.5) &= \exp[-0.025 \times 0.5] = 0.9876 \\
B(t, t+1) &= \exp[-0.5 \times (0.025 + 0.015)] = 0.9802 \\
B(t, t+1.5) &= \exp[-0.5 \times (0.025 + 0.015 + 0.041)] = 0.9603 \\
B(t, t+2) &= \exp[-0.5 \times (0.025 + 0.015 + 0.041 + 0.0389)] = 0.9418\,.
\end{aligned}
$$

Remark : Observe that we have to **convert annual** short forward rates on a semiannual basis, given that the **time step is the semester**.

- Frequently, in the asset pricing literature we focus on forward rates for future periods of **infinitesimal length**. The so-called **instantaneous forward rate** is defined as:

$$f(t,\tau) = \mathrm{lim}_{T\to\tau} R(t,\tau,T)\,,$$

and the function $\tau \mapsto f(t,\tau)$ is called the **term structure of instantaneous forward rates** or the **instantaneous forward rate curve**.

- Letting $T \to \tau$ in relation (13) we get by definition:

$$f(t,\tau) = -\frac{\partial \ln B(t,\tau)}{\partial \tau} = -\frac{\partial B(t,\tau)/\partial\tau}{B(t,\tau)}\,, \tag{8}$$

assuming a differentiable discounting function $B(t,\tau)$.

☐ From (8) we can also write:

$$B(t,\tau) = \text{exp}\left[-\int_t^\tau f(t,u)du\right] \,. \tag{9}$$

From (13) we can also deduce:

$$f(t,\tau) = \frac{\partial[R(t,\tau)(\tau - t)]}{\partial\tau} = R(t,\tau) + \frac{\partial R(t,\tau)}{\partial\tau}(\tau - t)\,,$$

and $f(t,\tau) - R(t,\tau)$ provides information about the slope of the yield curve. From (2) and (9) we see that:

$$R(t,\tau) = \frac{1}{\tau - t}\int_t^\tau f(t,u)du\,. \tag{10}$$

i.e. $R(t,\tau)$ is an average of instantaneous forward rates.

- $R(t,T)$ can be seen as a risk-free rate of return, over the fixed period $[t,T]$, induce by the investment in $B(t,T)$. In the continuous time literature, when we talk about **the risk-free rate**, or the **short rate** we mean the **instantaneous spot rate** defined as:

$$r(t) = \lim_{T \to t^+} R(t,T) = \lim_{T \to t^+} f(t,T)\,.$$

- The short rate $r(t)$ gives the possibility to introduce the notion of **bank account**: this (instantaneously) risk-free asset is a continuous roll-over of such instantaneously risk-free investments. Let $A = (A_t)$ denote the price process of the bank account.

☐ The increment to the balance of the bank account over an infinitesimal interval $[t, t+dt]$ is known at time t and given by

$$dA_t = A_t r(t) dt \,.$$

☐ An initial deposit of A_0 at time $t = 0$ grow at time t to:

$$A_t = A_0 \exp\left(\int_0^t r(u) du\right) \,.$$

☐ **Remark :** even if $r(t)$ is an (instantaneously) risk-free rate (over the time interval $[t, t+dt]$), its future values are not know at date t. For that reason, the process $(r(t))$ is not deterministic.

1.3.6 Forward Rates and Forward Discount Factors

- ☐ We have seen that the annually compounded spot rate $Y(t,T)$ set at date t concerns the price on a loan between the same date t (trading and settlement dates coincide) and the maturity date T.

- ☐ In the case of **forward rates**, the loan is received at some future settlement date $\tau \geq t$ and the maturity date is (as usual) $T > \tau \geq t$ ($t =$ trading date, $\tau =$ settlement date and $T =$ maturity date).

- ☐ In other words, this is the rate which is appropriate at time t for discounting between τ and T.

☐ The **annually compounded forward rate**, denoted $Y(t,\tau,T)$, with $t \le \tau < T$, is the rate such that:

$$(1+Y(t,T))^{-(T-t)} = (1+Y(t,\tau))^{-(\tau-t)} \times (1+Y(t,\tau,T))^{-(T-\tau)}, \qquad (11)$$

(when $t=\tau$ the forward rate reduces to the spot rate). We can also write (11) in terms of **forward discount factor** $F(t,\tau,T)$:

$$F(t,\tau,T) = \frac{B(t,T)}{B(t,\tau)} = \frac{1}{(1+Y(t,\tau,T))^{(T-\tau)}}. \qquad (12)$$

☐ The **forward discount factor** at time t defines the time value of money between two future dates, τ and $T > \tau$, and it is given by the ratio of the two date-t discount factors $B(t,\tau)$ and $B(t,T)$.

□ The forward discount factor has the following properties: $i)$ $F(t,\tau,T) = 1$ for $T = \tau$; $ii)$ $F(t,\tau,T)$ is decreasing in T.

□ The **forward rate** at time t for a risk-free investment from τ to T, and **with compounding frequency** m, is the interest rate determined by $F(t,\tau,T)$:

$$Y^{(m)}(t,\tau,T) = m \times \left(\frac{1}{F(t,\tau,T)^{\frac{1}{m\,(T-\tau)}}} - 1 \right).$$

□ The **continuously compounded forward rate** is obtained for $m \to +\infty$:

$$R(t,\tau,T) = -\frac{1}{(T-\tau)} \ln(F(t,\tau,T)).$$

□ Given an m-times compounded forward rate $Y^{(m)}(t,\tau,T)$, the discount factor is:

$$F(t,\tau,T) = \frac{1}{\left(1 + \frac{Y^{(m)}(t,\tau,T)}{m}\right)^{m\,(T-\tau)}}.$$

□ Given a continuously compounded forward rate $R(t,\tau,T)$, we have:

$$F(t,\tau,T) = \exp(-\,R(t,\tau,T)\,(T-\tau))\,.$$

□ If the discount factor $B(t,T)$ is increasing between two dates τ and $T > \tau$, that is $B(t,\tau) < B(t,T)$, then the forward rate at t for an investment between τ and T is negative. Nevertheless, $B(t,T)$ is decreasing in T.

□ The **forward curve** gives the relation between the forward rate $Y(t,\tau,T)$ and the time of the investment τ. It is also called the **Term Structure of forward rates**.

□ Both the Forward curve is derived from the same discount factors that determine the Spot Curve.

□ Indeed, we have that:

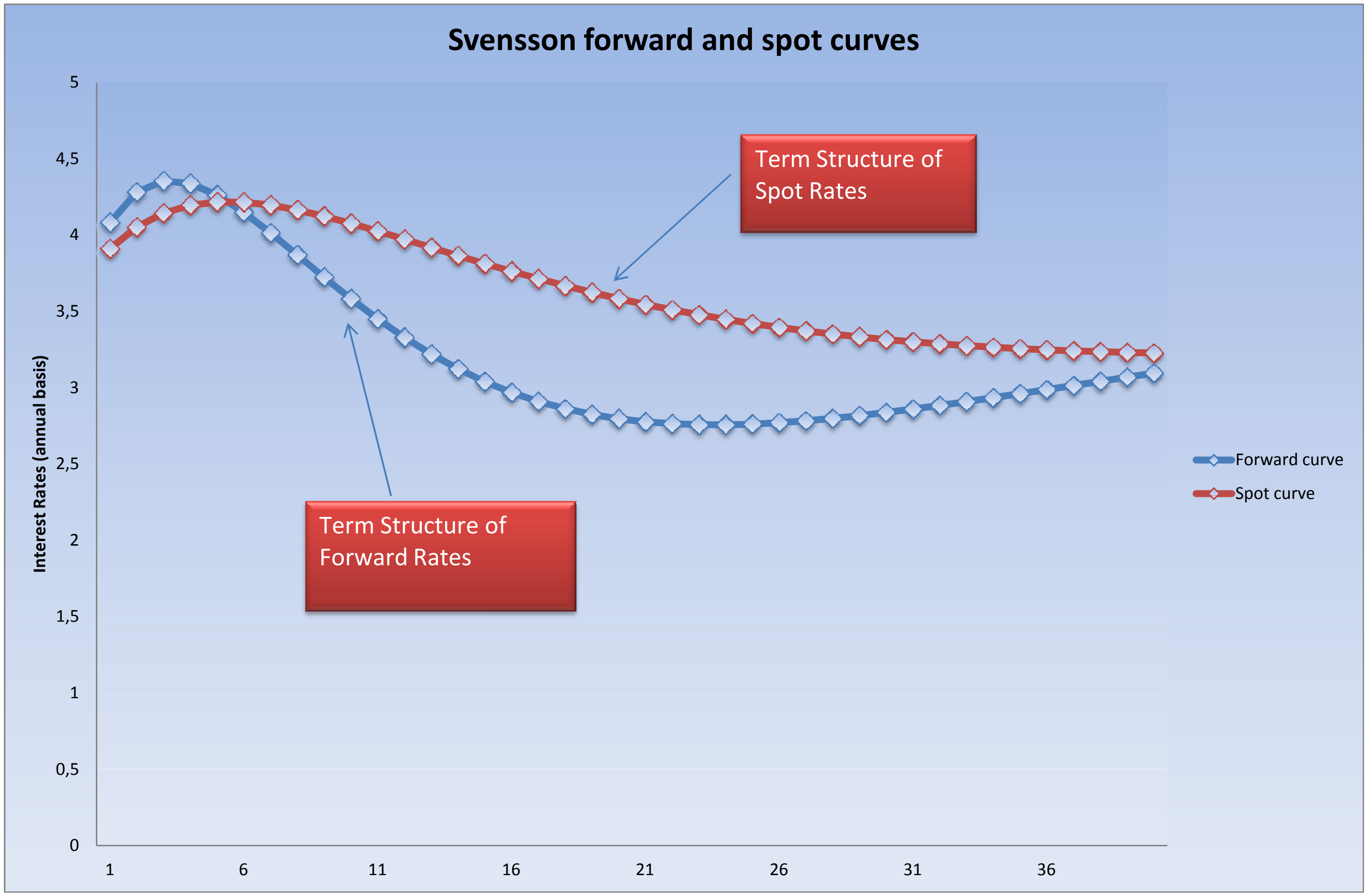
$$\begin{aligned} R(t,\tau,T) &= -\frac{1}{T-\tau}\ln\frac{B(t,T)}{B(t,\tau)} = \frac{R(t,T)(T-t)-R(t,\tau)(\tau-t)}{T-\tau} \\ &= R(t,\tau)+(T-t)\,\frac{R(t,T)-R(t,\tau)}{T-\tau}\,. \end{aligned} \tag{13}$$

☐ We find that, for any maturity τ, the forward rate between τ and T equals the spot rate $R(t, \tau)$ plus a term that is positive if the spot curve is rising, and it is negative if the spot curve is declining at τ. Thus, when the spot curve is

$i)$ **increasing** $\rightarrow$ the forward curve is **above** the spot curve.

$ii)$ **decreasing** $\rightarrow$ the forward curve is **below** the spot curve.

$iii)$ **flat** $\rightarrow$ the forward curve **equals** the spot curve.



Svensson forward and spot curves
Interest Rates (annual basis)
5
4,5
4
3,5
3
2,5
2
1,5
1
0,5
0
1
6
11
16
21
26
31
36
Term Structure of Spot Rates
Term Structure of Forward Rates
Forward curve
Spot curve

- We can also write, given the **short forward rate** $R(t, \tau, \tau+1) = \ln \frac{B(t,\tau)}{B(t,\tau+1)}$:

$$B(t,\tau) = \exp\left[-\sum_{j=t}^{\tau-1} R(t,j,j+1)\right] .$$

- and, thus

$$R(t,\tau) = \frac{1}{\tau - t} \sum_{j=t}^{\tau-1} R(t,j,j+1) .$$

☐ We have also seen that the **simply compounded forward (LIBOR) rate** at date t, valid for the period $[\tau, T]$, is the rate $L(t,\tau,T)$ such that:

$$B(t,\tau) = B(t,T) \times [1 + L(t,\tau,T) \times (T-\tau)] \, .$$

☐ The **simply compounded forward discount factor** $LF(t,\tau,T)$ is thus:

$$LF(t,\tau,T) = \frac{B(t,T)}{B(t,\tau)} = \frac{1}{[1 + L(t,\tau,T) \times (T-\tau)]} \, .$$

☐ and therefore

$$L(t,\tau,T) = \frac{1}{T-\tau}\left(\frac{B(t,\tau)}{B(t,T)} - 1\right) \, .$$

- Let us remember also that the **simply compounded spot (LIBOR) rate** $L(t,T)$ for the period $[t,T]$ is such that the present value of 1 unit of money paid at T is:

$$B_L(t,T) = \frac{1}{[1 + L(t,T) \times (T-t)]}\,,$$

- that is:

$$L(t,T) = \frac{1}{T-t}\left(\frac{1}{B_L(t,T)} - 1\right)\,.$$

1.3.7 Par yields

- It is a popular way to express yields on coupon-bearing bonds is by means of the concept of **par yield**.

- The **par yield** at date t for a given maturity date T, denoted $\rho(t,T)$ (say), is the coupon rate of a bullet bond making its price at date t equal to its par (face) value $(CB(0,T) = C_T)$.

- Even if zero-coupon yields are a more fundamental concept, par yields remain an alternative way to represent the yield curve. They are two alternative way to represent the information in the discount function.

- More formally, the par yield associated to a bullet bond with coupon dates $T_1, T_2, \dots, T_n$, coupon payments at a regular time interval of Δ (represented on annual basis), ZCB prices $B(t,T_1), \dots, B(t,T_n)$ and face value $C_T = 1$, **is the coupon rate** $\rho(t,T)$ (on annual basis) at which the asset would trade at par. This means that, imposing $CB(t,T) = C_T = 1$, $\rho(t,T)$ can be determined from:

$$1 = \sum_{i=1}^{n} 1_{\{t<T_i\}} \Delta \rho(t,T) B(t,T_i) + 1_{\{t<T_n\}} B(t,T_n).$$

- For simplicity, this formula assumes that a coupon payment has just been made, so that there is no accrued interest.

□ From this definition we find that $\rho(t,T)$ is given by:

$$\rho(t,T) = \frac{1}{\Delta} \times \frac{1 - B(t,T_n)}{\sum_{i=1}^{n} B(t,T_i)}, \quad t < T_1 .$$

and it reflects the date-t market interest rate for a bullet bond maturing at T.

Example : If we consider again the following data:

$\tau - t$	0.5	1	1.5	2
$B(t,\tau)$	0.9876	0.9802	0.9603	0.9418

then $\rho(t,t+2) = \dfrac{1}{0.5} \times \dfrac{1 - 0.9418}{0.9876 + 0.9802 + 0.9603 + 0.9418} = 0.03.$

1.3.8 Inflation-Indexed Bonds

- ☐ In this course we will mainly focus on bonds paying coupons and the final principal in dollars (say). In other words, the payments are *fixed in nominal terms*.

- ☐ Nevertheless, how much of a good an investor (a consumer) can buy with the *given* amount of dollars, obtained from the coupon and principal payments, depends on the *inflation* between the purchase of the bond and the coupon and principal payments.

- The CPI index, computed monthly by the Bureau of Labor Statistics (BLS), provides a weighted average of the value of a basket of representative goods that U.S. consumers purchase.

- The are different measures of the CPI, depending on the location and the type of goods considered. Here we consider the non-seasonally-adjusted U.S. City Average All Items Consumer Price Index ($CPI - U$), which is the index used for TIPS.

- The CPI variation over time measures the realized inflation (over that period) and it identifies the *inflation risk*.

☐ *Inflation risk* is the loss of purchasing power of the dollar. All assets that provide future payment in fixed nominal terms are affected by inflation risk.

☐ Let us show how are calculated the payments of **inflation-linked bonds**. We will focus, in particular, on TIPS (*Treasury Inflation Protected Securities*) traded in the U.S. bond market.

☐ TIPS are coupon bonds issued with maturities 5, 10 and 20 years. The coupon rate of TIPS is a constant fraction of the principal but the principal is not fixed: it changes over time to compensate for inflation.

☐ Let us consider a TIPS issued at date T_0 with annual (nominal) coupon rate of c_i, semiannual payments and face value $C_T = 100$. The coupon payment C_i^* (say) occurring at the month $m(T_i)$ of date $T_i \in \{T_1, \ldots, T_n\}$ is:

$$C_i^* = \left(\frac{c_i}{2} C_T\right) \times \frac{CPI^{ref}(m(T_i))}{CPI^{ref}(m(T_0))} \text{ , } \forall i \in \{1, \ldots, n\}$$

and the principal payment at $T_n = T$ will be:

$$C_T^* = C_T \times \max\left(\frac{CPI^{ref}(m(T_n))}{CPI^{ref}(m(T_0))}, 1\right) \text{ ,}$$

- where $CPI^{ref}(m(T_i))$ is called the *reference* CPI of date T_i and it is given by:

$$CPI^{ref}(m(T_i)) = CPI(m(T_i) - 2)\,\frac{d_i - 1}{d_n^i} + CPI(m(T_i) - 3)\,\frac{d_n^i - d_i + 1}{d_n^i}$$

- and where $CPI(t)$ is the CPI-U of month t (released typically during the third week), $d_i =$ day of payment in $m(T_i)$ and $d_n^i =$ number of days in $m(T_i)$.

□ The reason for the indexation lag is that the Bureau of Labor statistics publishes these data with a lag, with the index for a given month released in the middle of the subsequent month.

□ $Idx(T_0, T_i) = \dfrac{Idx(T_i)}{Idx(T_0)} = \dfrac{CPI^{ref}(m(T_i))}{CPI^{ref}(m(T_0))}$ is called the *Index Ratio* for the coupon C_i^* and it may be larger or smaller than one. This means that, in a deflationary period, coupon payments may reduce (contrary to principal payment).

☐ If the future inflation were known, the price at t of a coupon bond TIPS maturing at T would be simply given by the present value of future nominal payments:

$$
\begin{aligned}
CB^{TIPS}(t,T) &= \sum_{i=1}^{n} 1_{\{t<T_i\}} \left(\frac{c_i}{2} C_T\right) Idx(T_0,T_i)\, B(t,T_i) + C_T\, Idx(T_0,T_n)\, B(t,T_n) \\
&= \frac{1}{Idx(T_0)} \left[\sum_{i=1}^{n} 1_{\{t<T_i\}} \left(\frac{c_i}{2} C_T\right) Idx(T_i)\, B(t,T_i) + C_T\, Idx(T_n)\, B(t,T_n)\right]
\end{aligned}
$$

☐ In the case of a zero-coupon TIPS maturing at T and with face value C_T, we have:

$$
B^{TIPS}(t,T) = C_T \times \frac{Idx(T_n)}{Idx(T_0)} B(t,T)\,.
$$

☐ Let us define the **real Market Discount Factor** $B^{real}(t,T)$ as:

$$B^{real}(t,T) = Idx(t,T)\, B(t,T) = \frac{Idx(T)}{Idx(t)}\, B(t,T);$$

it is the exchange rate between consumption goods at t versus consumption goods at T; it is the date-t price of the real zero-coupon bond.

☐ The associated **real yield to maturity** is $R^{real}(t,T) = -\frac{\ln(B^{real}(t,T))}{T-t}$ and the **real term structure of interest rates** at time t is given by $R^{real}(t,T)$ with T varying.

☐ The date-t value (in consumption goods) of the real coupon bond maturing at T is given by:

$$CB^{real}(t,T) \;=\; \sum_{i=1}^{n} 1_{\{t<T_i\}} \left(\frac{c_i}{2}\,C_T\right) B^{real}(t,T_i) + C_T\,B^{real}(t,T_n)$$

☐ From these results we can easily write :

$$CB^{TIPS}(t,T) \;=\; \frac{Idx(t)}{Idx(T_0)} \left[\sum_{i=1}^{n} 1_{\{t<T_i\}} \left(\frac{c_i}{2}\,C_T\right) B^{real}(t,T_i) + C_T\,B^{real}(t,T_n)\right]$$

and, in particular, we have:

$$B^{TIPS}(t,T) \;=\; \frac{Idx(t)}{Idx(T_0)}\,C_T\,B^{real}(t,T_n)\,.$$

□ From $B^{real}(t,T) = Idx(t,T)\, B(t,T)$ we have:

$$\frac{Idx(t)}{Idx(T)} \exp[-\, R^{real}(t,T)\,(T-t)] = \exp[-\, R(t,T)\,(T-t)]\,.$$

□ Now, if we denote by π the **constant** continuously compounded annualized inflation rate between t and T, that is:

$$Idx(T) = Idx(t)\, \exp(\pi\,(T-t))$$

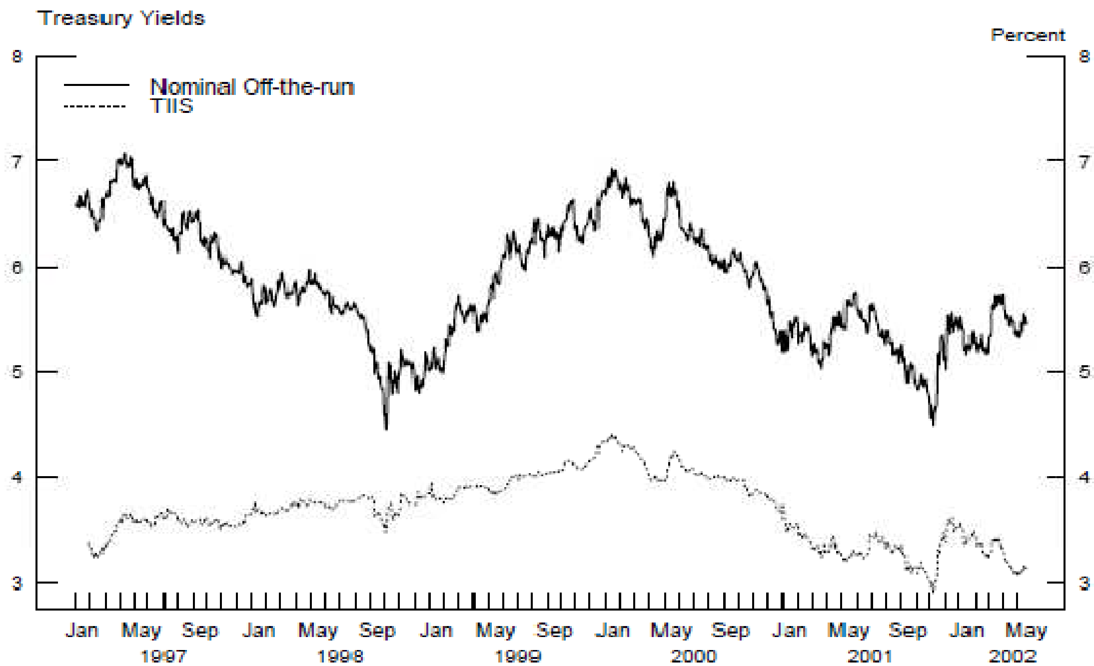
we immediately obtain $R(t,T) = R^{real}(t,T) + \pi$ (under perfect foresight!).

- In a general context where investors are risk-averse and there is uncertainty about future inflation, the quantity

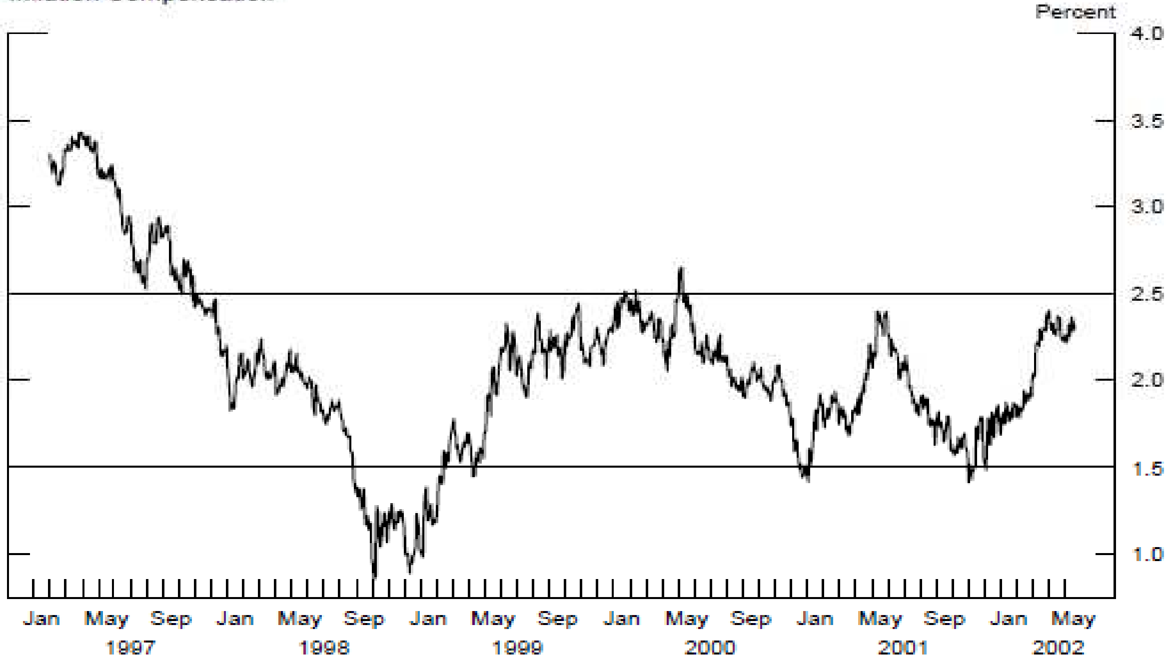
$$BEIR(t,T) = R(t,T) - R^{real}(t,T)$$

is called (spot) **inflation compensation** or **Break Even Inflation Rate** between t and T.

- Remember that the $BEIR$ IS NOT a measure of **inflation expectations** given that it contains also **inflation risk premium**; $BEIR$ is the sum of these two components. The latter component refers to the risk that realized inflation may deviate from the expected one.


Treasury Yields
Percent
Nominal Off-the-run
TIIS
8
7
6
5
4
3
Jan May Sep Jan May Sep Jan May Sep Jan May Sep Jan May Sep Jan May
1997 1998 1999 2000 2001 2002

Inflation Compensation



Data are through May 20, 2002.

- These two last pictures are taken from Sack and Elsasser (2004): "Treasury Inflation-Indexed Debt: A Review of the U.S. Experience" available at

  http://www.newyorkfed.org/research/epr/04v10n1/0405sack.pdf

- Nominal U.S. term structure of interest rates available at:

  http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html

- Real U.S. term structure of interest rates available at:

  http://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html