Fixed Income and Credit Risk
Winter 2013
Solutions for the Exam

January 30, 2013

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Exercise N° 01.

i) A Repurchase Agreement (Repo) is an agreement to sell (today) some securities to another party and buy them back at a fixed future date and for a fixed amount. The price at which the security is bought back is greater than the selling price and the difference implies an interest called Repo Rate. The repo can be seen as a collateralized loan and the advantage of this borrowing of funds is that the applied (Repo) rate is less than the cost of banking financing.

A Reverse Repo is the opposite transaction, namely, it is the purchase of the security for cash with the agreement to sell it back to the original owner at a predetermined price, determined, once again, by the Repo Rate. It is the same repurchase agreement from the repo dealer's viewpoint, not the trader's.

ii) The return on capital of the trader is:

$$\frac{P_T - P_t - \dfrac{1}{12} \times \textit{repo rate} \times (P_t - \textit{haircut})}{\textit{haircut}}$$

$$= \frac{99.01 - 98.5 - \dfrac{1}{12} \times 0.05 \times (98.5 - 0.8)}{0.8} = 0.13\,.$$

iii) The profit $\Pi(t,T)$ is:

$$P_t \times \left(1 + \frac{1}{52} \times \textit{repo rate}\right) - P_T = 99.40 \times \left(1 + \frac{1}{52} \times 0.06\right) - 99.48 = 0.0347\,.$$

Exercise N° 02.

i) Let us consider an incomplete market without redundant assets ($k = rank(\mathbf{S}') = d+1 < N$), a payoff $y \in \mathcal{M}(\mathbf{S})$ and the associated system $\mathbf{S}'\varphi = y$. Given that $rank(\mathbf{S}') = d+1$ we have that $(\mathbf{SS}')^{-1}$ exists and the right inverse of $\mathbf{S}$ is well defined : $R^{(S)} = \mathbf{S}'(\mathbf{SS}')^{-1}$. This means that we also have $(R^{(S)})' = (\mathbf{SS}')^{-1}\mathbf{S} = L^{(S')}$. Thus, the replicating strategy is $\varphi = L^{(S')}y = (\mathbf{SS}')^{-1}\mathbf{S}y$ which a solution of the original system being $y \in \mathcal{M}(\mathbf{S})$.

Now, the price of the payoff $y \in \mathcal{M}(\mathbf{S})$ is the value of the replicating portfolio an therefore $q(y) = S(0)'\varphi = S(0)'L^{(S')}y = S(0)'(\mathbf{SS}')^{-1}\mathbf{S}y = y'R^{(S)}S(0)$ and the pricing formula is proved.

ii) We have an incomplete market with redundant assets ($k < d+1$, $k < N$). Let us denote with $\bar{\mathbf{S}}$ the (k, N) payoff matrix of the no redundant assets and with $\bar{S}(0)$ the vector of these k asset prices. Given that $rank(\mathbf{S}') = k$ we have that $(\bar{\mathbf{S}}\bar{\mathbf{S}}')^{-1}$ exists and the right inverse of $\bar{\mathbf{S}}$ is well defined : $R^{(\bar{S})} = \bar{\mathbf{S}}'(\bar{\mathbf{S}}\bar{\mathbf{S}}')^{-1}$. This means that we also have $(R^{(\bar{S})})' = (\bar{\mathbf{S}}\bar{\mathbf{S}}')^{-1}\bar{\mathbf{S}} = L^{(\bar{S}')}$. Following the same steps as above, we find $q(y) = y'\,R^{(\bar{S})}\,\bar{S}(0) = \bar{S}(0)'\,L^{(\bar{S}')}\,y$ and the pricing formula is proved.

Exercise N° 03.

Given that $M_{t,t+1}$ is exponential-affine in ε_{t+1} (i.e. x_{t+1}) and that the conditional Laplace transform of x_{t+1} is exponential-affine in the conditioning variable (x_t) we suggest that the ZCB pricing formula at date t be an exponential-affine function of x_t and then "we check if it works". We proceed in the following way:

a) We suggest $B(t,t+h) = \exp(c_h'\,X_t + d_h)$ and we (equivalently) rewrite the pricing formula in terms of the payoff $B(t+1,t+h) = \exp(c_{h-1}'\,X_{t+1} + d_{h-1})$:

$$\begin{aligned} B(t,t+h) &= \exp(c_h'\,X_t + d_h) \\ &= E_t[M_{t,t+1}\cdots M_{t+h-1,t+h}] \\ &= E_t[M_{t,t+1}B(t+1,t+h)] \\ &= E_t\left[\exp\left(-\beta - \alpha'\,X_t + \Gamma_t\,\varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma_t^2\right)\exp(c_{h-1}'\,X_{t+1} + d_{h-1})\right]\,, \end{aligned}$$

b) we do the algebra (calculating the conditional Laplace transform) obtaining:

$$\begin{aligned} & B(t,t+h) \\ = \;& \exp(c_h{}'X_t + d_h) \\ = \;& \exp\left[-\beta - \alpha'X_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1}\right] \times E_t[\exp\left(\Gamma_t\varepsilon_{t+1} + c_{h-1}'X_{t+1}\right)] \\ = \;& \exp\left[-\beta - \alpha'X_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1} + c_{h-1}'(\Phi X_t + \tilde{\nu})\right] \times\; E_t[\exp\left(\Gamma_t + \sigma c_{1,h-1}\right)\varepsilon_{t+1})] \\ = \;& \exp\left[(-\alpha + \Phi' c_{h-1} + c_{1,h-1}\sigma\gamma)'X_t \right. \\ & \qquad \left. + \,(-\beta + c_{1,h-1}\nu + \tfrac{1}{2}c_{1,h-1}^2\sigma^2 + \gamma_o c_{1,h-1}\sigma + d_{h-1})\right]\,, \end{aligned}$$

c) and by identifying the coefficients we find the recursive relations for c_h and d_h characterizing the pricing formula $B(t,t+h) = \exp(c_h'\,X_t + d_h)$.

Now, the last elements we need to completely determine the pricing formula are the starting conditions for c_h and d_h. We proceed as follows:

given that, by definition of ZCB, we have $B(t,t) = 1$, then

$$\exp(c_0'\,X_t + d_0) = 1 \iff (c_0'\,X_t + d_0) = 0 \;\;\forall\;\; X_t \iff c_0 = 0\,,\;\; d_0 = 0\,.$$

We can also equivalently write:

given that, by definition of ZCB, we have $B(t,t+1) = \exp(-\,r_t)$, then

$$\exp(c_1'\,X_t + d_1) = \exp(-\,r_t) \iff (c_1'\,X_t + d_1) = -r_t \;\;\forall\;\; X_t \iff c_1 = -\,\alpha\,,\;\; d_1 = -\,\beta\,.$$

Exercise N° 04 [1 point].

Give the information we obtain from the market, we have that the one-quarter discount factor (i.e., the zero-coupon bond price for a unitary face value) is $B(0, 0.25) = 0.9880$. In addition, given the representation of the coupon bond price as a portfolio of zero-coupon bonds we can write:

$$100.960 = 0.9880 \times \frac{0.025}{4} \times 100 + B(0, 0.5) \times 100 \times (1 + 0.025/4)\,,$$

we easily obtain $B(0, 0.5) = 0.9972$. With these discount factors in hand, we can calculate the price of the floating rate bond thanks to the following formula:

$$\begin{aligned} CB_{FR}(0, 0.5) &= 100 + s \times 100 \times [B(0, 0.25) + B(0, 0.5)] \\ &= 100 + \frac{0.015}{4} \times 100 \times (0.9880 + 0.9972) = 100.744\,. \end{aligned}$$

Exercise N° 05.

The value of the swap is given by:

$$\begin{aligned} & V_{swap}(0, 1.5; 5.52\%) \\ = \;& 100 - \left[\sum_{j=0.5}^{1.5} \frac{0.0552}{2} \times 100 \times B(0, j) + 100 \times B(0, 1.5) \right] \\ = \;& 100 - \left[\frac{0.0552}{2} \times 100 \times 0.945 + \frac{0.0552}{2} \times 100 \times 0.990 + \left(\frac{0.0552}{2} + 1 \right) \times 100 \times 0.915 \right] \\ = \;& 100 - 2.6082 - 2.7324 - 94.0254 = 0.634\,, \end{aligned}$$

and therefore the swap rate $c = 5.52\%$ is not the correct one given that, at the inception of the contract, the swap value has to be equal to zero. The correct swap rate is:

$$\begin{aligned} c &= 2 \times \frac{1 - B(0, T_n)}{\sum_{j=1}^{n} B(0, T_j)} \\ &= 2 \times \frac{1 - 0.915}{0.945 + 0.990 + 0.915} = 0,05965\,. \end{aligned}$$

Exercise N° 06.

First, we need to solve for the default probability curves for the individual credits. Let $D(t)$ be the risk-free discount factor from time t to the present. Assume a common recovery value of R for each credit. Set up the CDS pricing equation:

$$\text{(buyer)} \qquad \bar{s}\sum_{t=1}^{5}(1-P(t))D(t) = \sum_{t=1}^{5}(P(t)-P(t-1))(1-R)(1+c)D(t) \qquad \text{(seller)}$$

Using the constant hazard rate model, we plug in $P(t) = 1-\exp[-\lambda t]$, and solve () numerically for λ given the fair spread $\bar{s}$.

Now define $L(t)$ to be the cumulative losses on the pool up to time t. Then the bond notional at time t is:

$$N(t) = 1 - \max\{L(t)-a, 0\} = \min\{1, 1+a-L(t)\}.$$

The price of the bond is the expectation of its discounted interest and principal cash-flows

$$c \cdot \sum_{t=1}^{T} N(t-1)D(t) + N(T)\cdot D(T),$$

with $T = 5$. To complete the pricing, we need an expression for the expectation of $N(t)$, for $t = 1, \ldots, T$. Let ρ be the common asset correlation across the credits in the pool. Since the pool is large and homogeneous, we may model using the fine-grained limit. Thus $L(t)$ is equal in distribution to

$$(1-R)\cdot\Phi\left(\frac{\alpha(t)-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right),$$

where Z is standard normal distributed, and $\alpha(t) = \Phi^{-1}(P(t))$. Let ϕ denote the p.d.f. for the standard normal distribution. The expectation is then

$$\begin{aligned} lll\mathbf{E}N(t) &= \mathbf{E}\min\{1, 1+a-L(t)\} \\ &= \int_{-\infty}^{\infty} dz\phi(z)\min\left\{1, 1+a-(1-R)\Phi\left(\frac{\alpha(t)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)\right\}. \end{aligned}$$

In order to simplify the integral, we solve

$$(1-R)\cdot\Phi\left(\frac{\alpha(t)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) = a,$$

for z. The solution is given by

$$z^\star = \frac{\alpha(t)-\sqrt{1-\rho}\Phi^{-1}(a/(1-R))}{\sqrt{\rho}}.$$

So we have

$$(1-R)\cdot\Phi\left(\frac{\alpha(t)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right) \geq a, \text{ if } z \leq z^\star,$$

and vice-versa. We can then simplify the integral

$$\begin{aligned}
\mathbf{E}N(t) &= \int_{-\infty}^{z^\star} dz\phi(z)\left[1+a-(1-R)\Phi\left(\frac{\alpha(t)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)\right]+\int_{z^\star}^{\infty} dz\phi(z) \\
&= (1+a)\Phi(z^\star)+(1-\Phi(z^\star))-(1-R)\int_{-\infty}^{z^\star} dz\phi(z)\Phi\left(\frac{\alpha(t)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right).
\end{aligned}$$

The last term must be evaluated numerically.